DEFINABILITY IN MONOIDAL ADDITIVE AND TENSOR TRIANGULATED CATEGORIES

A thesis submitted to The University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2023

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Contents

Word Count: 33,927

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The aim of this thesis is to investigate definability in monoidal additive categories. Given a monoidal finitely accessible category $\mathcal{C}$, satisfying certain assumptions, we prove that there exists an inclusion-reversing bijection between the fp-hom-closed definable subcategories of $\mathcal{C}$ and the Serre tensor-ideals $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}$-mod. We use this result to prove that the 2-category of skeletally small abelian categories with additive exact symmetric monoidal structures is anti-equivalent to the 2-category of fp-hom-closed definable additive categories satisfying an exactness criterion. We define a Ziegler-type topology, $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$, whose closed subsets correspond to the fp-hom-closed definable subcategories of $\mathcal{C}$. We demonstrate that, in general, $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ is non-trivial, distinct from $\mathrm{Zg}(\mathcal{C})$ and the topology on $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ depends on the monoidal structure on $\mathcal{C}$.

Under the additional assumption that $\mathcal{C}^{\mathrm{fp}}$ is a rigid monoidal subcategory of $\mathcal{C}$, we show that a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$ is fp-hom-closed if and only if it is a tensor-ideal. Furthermore, given $\mathcal{A}$, a small preadditive category with an additive symmetric rigid monoidal structure, we show that elementary duality maps an fp-hom-closed definable subcategory $\mathcal{D} \subseteq$ Mod-$\mathcal{A}$ to a definable tensor-ideal $\mathcal{D}^d \subseteq \mathcal{A}$-Mod and vice versa.

Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category. We provide tensor-analogues of Krause's *Fundamental Correspondence* between definable subcategories, Serre subcategories, cohomological ideals and closed subsets of the Ziegler topology, considering both $\mathcal{T}$-tensor-closed definable subcategories and definable tensor-ideals.

We explore connections between definable tensor-ideals and smashing subcategories, and define four new Ziegler-type topologies.

Finally, we define an internal tensor-duality on the definable subcategories of $\mathcal{T}$ and describe the resulting lattice isomorphisms between our Ziegler-type topologies and bijections between certain torsion-torsion-free triples in $\mathcal{T}$.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgements

First and foremost I would like to thank my supervisor Prof. Mike Prest for all his patience, wisdom and support. My PhD would not have been such a relaxed and enjoyable experience without him. I would also like to thank Prof. Peter Symonds for taking over as my official supervisor upon Mike's retirement.

In addition, I would like to thank Mike's other PhD students Rosie, Sam, Mike, Harry, Isaac and Sunny, as well as my other office mates, Ray, Deacon, Gabriel, Alex and Shi for interesting mathematical discussions, as well as Mahah, Sunny, Ray, Clément, Dan and Brian for all the strictly non-mathematical discussions and Zoom calls. I am also grateful to Louise Walker for the yoga sessions and the University of Manchester Dance Society for all the dance classes.

Thanks go to my friends and family for your unconditional love and support, in particular to Sam for all the pep talks and cups of tea.

Finally, I would like to thank the EPSRC and the University of Manchester for funding my PhD.

Chapter 1

Introduction

This thesis is concerned with exploring interactions between definability and monoidal structures. The thesis is divided into two parts. The first part (Chapters 3 and 4) is from the preprint [59] which is under review to be published and concerns definable subcategories of monoidal finitely accessible categories. The second part (Chapters 5, 6, 7 and 8) is based in the triangulated setting where we consider definability in rigidly-compactly generated tensor triangulated categories.

Finitely accessible additive categories are big categories (in the sense that they have a proper class of objects) which are generated by taking direct limits of a skeletally small subcategory of so-called finitely presentable objects. A key example of a finitely accessible category is the category of modules over some ring (or ring with many objects), studied in infinite-dimensional representation theory. One way to better understand the structure or complexity of a module category is to understand its definable subcategories. Born out of research into the model theory of modules, the definable subcategories of a finitely accessible category with products, $\mathcal{C}$, are those axiomatised by certain sentences called pp-pairs in a many-sorted language, $\mathscr{L}(\mathcal{C})$, associated to the category $\mathcal{C}$. Although $\mathcal{C}$ has a proper class of objects, the definable subcategories of $\mathcal{C}$ form a set. Definable subcategories are generated as such by certain objects called pure-injectives. The isomorphism classes of indecomposable pure-injectives in $\mathcal{C}$ form a set which underlies a topological space called the Ziegler spectrum. The Cantor-Bendixson rank of the Ziegler spectrum provides us with an invariant which sheds light on the complexity of the module category.

Although the motivation behind definable subcategories was model theoretic, they also have 'nice' algebraic properties. Indeed, a full subcategory of a finitely accessible category with products, $\mathcal{C}$, is definable if and only if it is closed under direct products, direct limits and pure subobjects. In addition, equivalence classes of pp-pairs in the language $\mathscr{L}(\mathcal{C})$ form the objects of a category, $\mathbb{L}_{\mathcal{C}}^{\mathrm{eq}+}$, which is equivalent to the functor category $\mathcal{C}^{\mathrm{fp}}$-mod of finitely presented additive functors from the category of finitely presentable objects of $\mathcal{C}$ to the category of abelian groups.

For a functor F in $\mathcal{C}^{\mathrm{fp}}$-mod, denote by $\overrightarrow{F} : \mathcal{C} \to \mathbf{Ab}$ the unique (up to isomorphism) extension of F to $\mathcal{C}$ which commutes with direct limits. Given a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$, the full subcategory $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}$-mod, consisting of all functors F such that $\overrightarrow{F}(X) = 0$ for all $X \in \mathcal{D}$, is a Serre subcategory (e.g. see [48, Theorem 12.4.1 and Corollary 12.4.2]). Consequently, $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ is a skeletally small abelian category. Suppose $\mathcal{D}$ is equivalent to a definable subcategory of $\mathcal{C}'$, with associated Serre subcategory $\mathsf{S}' \subseteq \mathcal{C}'$-mod. Then

$$\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S} \simeq \mathcal{C}'^{\mathrm{fp}}\text{-mod}/\mathsf{S}' \simeq (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$$

where $(\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ denotes the category of additive functors from $\mathcal{D}$ to the category of abelian groups which commute with direct products and direct limits ([49, Theorem 12.10], [34, Theorem 7.2] for the case $\mathcal{D} = \mathcal{C}$). We say that a category $\mathcal{D}$ is a definable category if there exists some finitely accessible category with products $\mathcal{C}$ such that $\mathcal{D}$ is equivalent to a definable subcategory of $\mathcal{C}$. In addition, we define $\mathbb{DEF}$ to be the 2-category with definable categories as objects, 1-morphisms given by additive functors which commute with direct products and direct limits and 2-morphisms given by natural transformations.

In [51], Prest and Rajani show that the assignment $\mathcal{D} \mapsto \mathrm{fun}(\mathcal{D}) := (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ extends to an anti-equivalence between the 2-category $\mathbb{DEF}$ and the 2-category $\mathbb{ABEX}$ with objects given by skeletally small abelian categories, 1-morphisms given by additive exact functors and 2-morphisms given by natural transformations.

Many key examples of finitely accessible categories studied in additive representation theory have an additive symmetric monoidal structure; for example the category of modules over a commutative ring or group algebra for a finite group. Given an additive symmetric monoidal structure on a finitely accessible category with products $\mathcal{C}$, we can induce a monoidal structure on the associated functor category $\mathcal{C}^{\mathrm{fp}}$-mod by Day convolution product (see [20] and Section 2.2). More generally, if $\mathcal{D} \subseteq \mathcal{C}$ is a definable subcategory and the associated Serre subcategory $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}$-mod is a tensor-ideal with respect to the induced monoidal structure, then the localisation $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S} \simeq \mathrm{fun}(\mathcal{D})$ inherits a monoidal structure (see Definition 3.3.9).

Assume that $\mathcal{C}$ is a finitely accessible category with products and a closed symmetric monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ is a closed monoidal subcategory. In addition, suppose that $\mathcal{D}$ is a definable subcategory of $\mathcal{C}$ with associated Serre subcategory $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}$-mod. In Chapter 3 we show that S is a Serre tensor-ideal if and only if $\mathcal{D}$ is fp-hom-closed, that is for all $A \in \mathcal{C}^{\mathrm{fp}}$ and all $X \in \mathcal{D}$, $\mathrm{hom}(A, X) \in \mathcal{D}$, where hom denotes the internal hom-functor (Theorem 3.3.6). Using this result, we define a 2-category $\mathbb{DEF}^{\otimes}$ with objects given by triples $(\mathcal{D}, \mathcal{C}, \otimes)$ where $(\mathcal{C}, \otimes)$ is a monoidal finitely accessible category satisfying the assumptions given above and $\mathcal{D}$ is an fp-hom-closed definable subcategory of $\mathcal{C}$ which satisfies an exactness criterion. The 1-morphisms of $\mathbb{DEF}^{\otimes}$ are the additive functors $I : \mathcal{D} \to \mathcal{D}'$ which commute with direct products and direct limits and such that the induced functor $I_0 : \mathrm{fun}(\mathcal{D}') \to \mathrm{fun}(\mathcal{D})$ (see [51, Theorem 2.3]) is monoidal and the 2-morphisms are given by natural transformations. Let $\mathbb{ABEX}^{\otimes}$ denote the 2-category with objects the skeletally small abelian categories equipped with an additive symmetric monoidal structure which is exact in each variable, 1-morphisms being the additive exact monoidal functors and 2-morphisms the natural transformations. Chapter 3 is dedicated to proving that the 2-categories $\mathbb{ABEX}^{\otimes}$ and $\mathbb{DEF}^{\otimes}$ are anti-equivalent.

Suppose we have a triple $(\mathcal{D}, \mathcal{C}, \otimes)$ where $\mathcal{D} \subseteq \mathcal{C}$ is an fp-hom-closed definable subcategory and $\mathcal{C}$ is a finitely accessible category with products and an additive symmetric monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ forms a monoidal subcategory. $(\mathcal{D}, \mathcal{C}, \otimes)$ is an object of $\mathbb{DEF}^{\otimes}$ if and only if the definable subcategory $\mathcal{D}$ satisfies an exactness criterion. If $\mathcal{D} \subseteq \mathcal{C}$ is an fp-hom-closed definable subcategory, then $\mathcal{D}$ and $\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab})$ are equivalent in $\mathbb{DEF}$. The exactness criterion is necessary to ensure that $\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab}) \subseteq \mathrm{fun}(\mathcal{D})$-Mod is also fp-hom-closed (Theorem 3.4.2 and Proposition 3.3.11). In practice, many fp-hom-closed definable subcategories do not satisfy the exactness criterion. Indeed, the exactness criterion for $\mathcal{D}$ implies that the monoidal structure on $\mathrm{fun}(\mathcal{D})$ is exact (Theorem 3.3.10), when in general this monoidal structure is only right exact.

In Chapter 4, we discuss the relationship between definability and monoidal structures for fixed $\mathcal{C}$. We define a coarser version of the Ziegler spectrum, denoted by $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$, on the set of (isomorphism classes of) indecomposable pure-injectives in $\mathcal{C}$ called the fp-hom-closed Ziegler topology (Section 4.1). $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ is defined such that there exists a lattice isomorphism between the lattice of closed subsets of $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ and the lattice of fp-hom-closed definable subcategories of $\mathcal{C}$. For $\mathcal{C} = R$-Mod, where R is a commutative ring, we provide an example showing that in general $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ is non-trivial and $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ can be different to $\mathrm{Zg}(\mathcal{C})$. In addition, we demonstrate that $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$ depends on the monoidal structure on $\mathcal{C}$, using two examples from [50, Section 13].

We also consider what can be said under the additional assumption that $\mathcal{C}^{\mathrm{fp}}$ is a rigid monoidal subcategory of $\mathcal{C}$ (Section 4.3). A monoidal category is said to be rigid if every object has a dual object. For example if $\mathcal{C} = kG$-Mod where G is a finite group and the tensor product is given by $\otimes_k$, then $\mathcal{C}^{\mathrm{fp}} = kG$-mod is a rigid monoidal category, where the dual of a module M is given by $\mathrm{Hom}_k(M, k)$. In this setting, a definable subcategory is fp-hom-closed if and only if it is a tensor-ideal (Corollary 4.3.1).

Given a skeletally small preadditive category $\mathcal{A}$, one can define the language for right $\mathcal{A}$-modules, $\mathscr{L}_{\mathcal{A}}$ and the language for left $\mathcal{A}$-modules, ${}_{\mathcal{A}}\mathscr{L}$. Elementary duality of pp formulas maps a pp formula ϕ in the language $\mathscr{L}_{\mathcal{A}}$ to a pp formula $D\phi$ in the language ${}_{\mathcal{A}}\mathscr{L}$ and vice versa. Elementary duality extends to pp-pairs and therefore can also be viewed as a duality between the functor categories $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ and $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. Elementary duality of pp-pairs, or equivalently finitely presented functors, gives rise to an elementary duality of definable subcategories. In Section 4.4, we show that if $\mathcal{A}$ has an additive, symmetric, rigid monoidal structure, then elementary duality yields a bijection between the fp-hom-closed definable subcategories of Mod-$\mathcal{A}$ and the definable tensor-ideals of $\mathcal{A}$-Mod.

The second part of this thesis is concerned with definability in rigidly-compactly generated tensor triangulated categories. In representation theory, triangulated categories are used to better understand the structure of important abelian categories. For example the derived category of a module category is the appropriate setting for homological algebra and the stable module category of a group algebra allows us to factor out the well-understood projective modules and focus on the non-projectives. The complexity of the stable module category indicates how far the module category is from being semisimple. In these key examples (provided we consider modules over a commutative ring) the triangulated category has a rigidly-compactly generated tensor triangulated structure.

As in the finitely accessible case, we can define a language, $\mathscr{L}(\mathcal{T})$, and use pp-pairs in this language to define the definable subcategories of a compactly generated triangulated category $\mathcal{T}$. What's more the set of (isomorphism classes of) indecomposable pure-injective objects in $\mathcal{T}$ form the underlying set of a topology, $\mathrm{Zg}_{\mathcal{T}}$, called the Ziegler spectrum of $\mathcal{T}$. These analogous definitions in the triangulated setting make sense and interact much like their finitely accessible counterparts. The reason behind these similarities is a strong connection between $\mathcal{T}$ and a definable subcategory of the finitely accessible category Mod-$\mathcal{T}^c$ of right $\mathcal{T}^c$-modules, namely the definable subcategory Abs-$\mathcal{T}^c$ of absolutely pure right $\mathcal{T}^c$-modules. Indeed, the restricted Yoneda functor $H : \mathcal{T} \to \text{Mod-}\mathcal{T}^c$, given by $X \mapsto H_X := (-, X)|_{\mathcal{T}^c}$, induces a homeomorphism between $\mathrm{Zg}_{\mathcal{T}}$ and $\mathrm{Zg}(\text{Abs-}\mathcal{T}^c)$ (and in particular an isomorphism between the lattice of definable subcategories of $\mathcal{T}$ and the lattice of definable subcategories of Abs-$\mathcal{T}^c$) [6, Theorem 1.10]. Furthermore the language for $\mathcal{T}$ is the language for right $\mathcal{T}^c$-modules and every $X \in \mathcal{T}$ becomes an $\mathscr{L}(\mathcal{T})$-structure in the same way that H_X can be viewed as a structure for the language of right $\mathcal{T}^c$-modules.

In 2002, Krause described a fundamental correspondence, [36], between definable subcategories, coherent functors, cohomological ideals and closed subsets of the Ziegler spectrum in the setting of compactly-generated triangulated categories (see Theorem 2.5.11). Here, any coherent functor can be realised as the assignment $X \mapsto \phi(X)$ for a pp formula ϕ in the language $\mathscr{L}(\mathcal{T})$ and any pp formula ϕ defines a coherent functor in this way [27, Lemma 4.3]. The correspondence between definable subcategories, coherent functors and closed subsets of the Ziegler spectrum echoes a result in the finitely accessible setting. The link to cohomological ideals in the triangulated case is a consequence of the following. Every pp formula in $\mathscr{L}(\mathcal{T})$ is equivalent to a division formula, ϕ_f defined to be $\exists y_B \; x_A = y_B f$ for some morphism $f : A \to B$ in $\mathcal{T}^c$ [27, Proposition 3.1]. The meaning of *equivalent* here will be made precise in Section 2.5. The cohomological ideal $\mathcal{J}$ corresponding to a definable subcategory $\mathcal{D}$ is given by all the morphisms f in $\mathcal{T}^c$ such that $\phi_f(X) = 0$ for all $X \in \mathcal{D}$.

In Chapter 6, we define a new Ziegler-type topology which we call the $\mathcal{T}$-tensor-closed Ziegler topology and provide a tensor-analogue of Krause's Fundamental Correspondence. Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category. In Theorem 5.1.8 and Proposition 6.1.13 we establish a inclusion-preserving bijective correspondence between the

(i) $\mathcal{T}$-tensor-closed definable subcategories $\mathcal{D} \subseteq \mathcal{T}$, which are the same as the

(ii) $\mathcal{T}^c$-tensor-closed definable subcategories $\mathcal{D} \subseteq \mathcal{T}$ and the

(iii) closed subsets of the $\mathcal{T}$-tensor-closed Ziegler topology $\mathscr{C} \subseteq \mathrm{Zg}^{\otimes}_{\mathcal{T}}$

and an inclusion-preserving bijective correspondence between the

(iv) Serre tensor-ideals $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$,

(v) Serre tensor-ideals $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ and the

(vi) $\mathcal{T}^c$-tensor-closed cohomological ideals $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$.

Furthermore, we show that (i)-(iii) correspond via inclusion-reversing bijections to (iv)-(vi). For undefined notation and terminology see Section 5.1.

Later work by Krause, ([37]), provides a restriction of the Fundamental Correspondence to the case where the definable subcategory, $\mathcal{D}$, is triangulated. Here we give a tensor-analogue of this restriction. For $\mathcal{T}$ a rigidly-compactly generated tensor triangulated category, we prove in Theorem 5.2.14 that the above tensor-analogue of Krause's Fundamental Correspondence restricts to a bijective correspondence between the

(i) definable tensor-ideals $\mathcal{D} \subseteq \mathcal{T}$,

(ii) smashing tensor-ideals $\mathcal{B} \subseteq \mathcal{T}$,

(iii) perfect Serre tensor-ideals $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ and the

(iv) $\mathcal{T}^c$-tensor-closed exact ideals $\mathcal{J} \subseteq \text{morph}(\mathcal{T}^c)$.

For undefined notation and terminology see Section 5.2.

In Chapter 6 we explore various topological spaces which can be associated to a rigidly-compactly generated tensor triangulated category $\mathcal{T}$. In Section 6.1 we define four different Ziegler-type topologies (including the one previously mentioned), namely the positive shift-closed Ziegler topology, $\text{Zg}_{\mathcal{T}}^{\Sigma^+}$, the negative shift-closed Ziegler topology $\text{Zg}_{\mathcal{T}}^{\Sigma^-}$, the shift-closed Ziegler topology $\text{Zg}_{\mathcal{T}}^{\Sigma}$ and the $\mathcal{T}$-tensor-closed Ziegler topology $\text{Zg}_{\mathcal{T}}^{\otimes}$. Let $\mathbb{O}(X)$ denote the lattice of open subsets of the topological space X. We have the following relationships between the lattices of open subsets

$$\mathbb{O}(\text{Zg}_{\mathcal{T}}^{\Sigma^+}) \cap \mathbb{O}(\text{Zg}_{\mathcal{T}}^{\Sigma^-}) = \mathbb{O}(\text{Zg}_{\mathcal{T}}^{\Sigma})$$

and

$$\mathbb{O}(\text{Zg}_{\mathcal{T}}^{\otimes}) \subseteq \mathbb{O}(\text{Zg}_{\mathcal{T}}^{\Sigma}) \subseteq \mathbb{O}(\text{Zg}_{\mathcal{T}}).$$

We show that the lattice of open subsets of the shift-closed Ziegler topology is isomorphic to the lattice of open subsets of a quotient topology of the Ziegler topology, but such an isomorphism does not exist for the $\mathcal{T}$-tensor-closed Ziegler topology (Proposition 6.1.10 and Example 6.1.20).

In Section 6.2 we consider the spectrum of a small tensor triangulated category $\mathcal{K}$, defined by Balmer in 2005, which here we call the Balmer spectrum of $\mathcal{K}$ and denote by $\text{Spc}(\mathcal{K})$ (see [8]). Inspired by the prime spectrum of a commutative ring, $\text{Spc}(\mathcal{K})$ has underlying set given by the so-called prime tensor-ideals of $\mathcal{K}$.

In Chapter 7, we use a result from [27] and our rigidity assumption to define an internal tensor-duality on the definable subcategories of $\mathcal{T}$. In Theorem 7.2.5 we prove that internal tensor-duality induces a lattice automorphism on $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}})$ which gives an isomorphism $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^+}) \cong \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^-})$, restricts to an automorphism on $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma}) = \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^+}) \cap \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^-})$ and fixes $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\otimes})$.

In [4], the authors establish a 1-1 correspondence between the compactly-generated TTF triples in $D(\text{Mod-}R)$ and the compactly-generated TTF triples in $D(R\text{-Mod})$ for any ring R [4, Theorem 3.1]. Notice that a compactly-generated TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ in either of these categories has definable middle spot $\mathcal{V}$. In the case that R is commutative, it can easily be seen that the 1-1 correspondence of [4, Theorem 3.1] is induced by the internal tensor-duality defined in Chapter 7. In Chapter 8, we generalise this result to algebraic rigidly-compactly generated tensor triangulated categories. More generally, we show that, given an algebraic rigidly-compactly generated tensor triangulated category $\mathcal{T}$, internal tensor-duality induces a bijective correspondence between the suspended TTF triples $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ such that $\mathcal{V}$ is definable and the cosuspended TTF triples $(\mathcal{U}', \mathcal{V}', \mathcal{W}')$ with $\mathcal{V}'$ definable (Theorem 8.1.16). This bijection restricts to a bijective correspondence between compactly generated suspended TTF triples and compactly generated cosuspended TTF triples (Proposition 8.1.11) and restrict to an automorphism on the class of all stable TTF triples $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ such that $\mathcal{V}$ is definable.

When $\mathcal{T} = D(\text{Mod-}R)$ where R is a commutative ring, the above correspondence between suspended and cosuspended TTF triples yields an injective map

$$\left\{ \begin{array}{c} \text{Silting objects } S \text{ in } D(\text{Mod-}R) \\ \text{with } S^{\perp>0} \text{ definable, up to equivalence} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{Pure−injective cosilting objects} \\ \text{in } D(\text{Mod-}R)\text{, up to equivalence} \end{array} \right\}.$$

The restriction of the above to the compactly generated case results in the injective map

$$\left\{ \begin{array}{c} \text{Silting objects of finite type} \\ \text{in } D(\text{Mod-}R)\text{, up to equivalence} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{Cosilting objects of finite type} \\ \text{in } D(R\text{-Mod})\text{, up to equivalence} \end{array} \right\}$$

from [4, Theorem 3.3], which the authors use to describe a silting-cosilting duality on bounded silting and cosilting complexes.

In Section 8.3 we consider the case $\mathcal{T} = D(R\text{-Mod})$ where R is a coherent commutative ring of weak global dimension at most one such that every finitely presented R-module has finite projective dimension. In this setting $\mathcal{T}$ comes equipped with the standard t-structure which allows us to use homological algebra to make comparisons between definability in $\mathcal{T}$ and its Grothendieck monoidal heart, Mod-R. In Proposition 8.3.9 we show that the nth cohomology of a $\mathcal{T}$-tensor-closed definable subcategory is fp-hom-closed. Let $\mathcal{D}$ be a definable subcategory of $\mathcal{T} = D(R\text{-Mod})$ with internal tensor-dual $\mathcal{D}^\vee$. In Theorem 8.3.13 we show that, for each $n \in \mathbb{Z}$, $H^n(\mathcal{D})$ and $H^n(\mathcal{D}^\vee)$ are elementary dual definable subcategories.

Chapter 2

Background

In this chapter we provide some background material which will be useful throughout the rest of the thesis. Sections 2.2, 2.3 and 2.4 from this chapter are largely based on material from [59].

2.1 Conventions and Notation

We assume basic knowledge of additive category theory. All our categories are preadditive and have small hom-sets, all functors are additive and all subcategories are closed under isomorphism. Given preadditive categories $\mathcal{A}$ and $\mathcal{B}$, we will denote by $(\mathcal{A}, \mathcal{B})$ the functor category of all additive functors from $\mathcal{A}$ to $\mathcal{B}$. The functor category $(\mathcal{A}, \mathbf{Ab})$ will be denoted by $\mathcal{A}$-Mod and the subcategory of all finitely presented objects with be denoted by $\mathcal{A}\text{-mod} := (\mathcal{A}\text{-Mod})^{\mathrm{fp}}$. Similarly, we denote by Mod-$\mathcal{A}$ and mod-$\mathcal{A}$ the categories $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})$ and $(\mathcal{A}^{\mathrm{op}}, \mathbf{Ab})^{\mathrm{fp}}$ respectively. For a preadditive category $\mathscr{C}$ we denote by $\mathscr{C}(X, Y)$ the abelian group of all morphisms in $\mathscr{C}$ from X to Y. When $\mathscr{C} = \mathcal{A}$-Mod or Mod-$\mathcal{A}$ we may denote the hom-set by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ and when the category is clear from context we will simply write (X, Y).

Many of the category theoretic results in this thesis are established up to equivalence of categories. We treat isomorphic objects as the same and blur the line between isomorphism and equality. On occasion we may identify a category with its skeleton. In addition, we often define an object up to isomorphism as a limit or colimit. Since our main application is representation theory, in which isomorphic modules are thought of as the same, these conventions are appropriate. We will use $\cong$ to denote isomorphism (usually between objects of a category) and $\simeq$ to denote equivalence. We will say there is a duality between categories $\mathscr{A}$ and $\mathscr{B}$ if there is an equivalence $\mathscr{A}^{\mathrm{op}} \simeq \mathscr{B}$. If $\lambda : \mathscr{A} \to \mathscr{B}$ is left adjoint to $\rho : \mathscr{B} \to \mathscr{A}$, we use the notation $\lambda \dashv \rho$.

Given a category $\mathscr{C}$, we will write $X \in \mathscr{C}$ to assert that X is an object of the category $\mathscr{C}$. We will write $f \in \mathrm{morph}(\mathscr{C})$ to mean 'f is a morphism in $\mathscr{C}$'. Similarly if $\mathcal{X}$ is a subcategory of $\mathscr{C}$ we will write $\mathcal{X} \subseteq \mathscr{C}$. We use this set theoretic language irrespective of whether $\mathscr{C}$ has a proper class of objects or not. Unless mentioned otherwise, all subcategories will be full subcategories and we may identify a class of objects with the full subcategory it determines. Given a 2-category $\mathscr{C}$, we will write $\mathscr{C}^{\mathrm{op}}$ to denote the category in which 1-morphisms are reversed but 2-morphisms are not reversed. Given an appropriate category $\mathscr{C}$ and a set of objects $\mathcal{X} \subseteq \mathscr{C}$, we write $\langle \mathcal{X} \rangle^{\mathrm{def}}$ to denote the smallest definable subcategory containing $\mathcal{X}$ and $\langle \mathcal{X} \rangle^{\mathsf{S}}$ to denote the smallest Serre subcategory containing $\mathcal{X}$. If $\mathcal{X}$ only has one object, say X, we will write $\langle X \rangle^{*}$.

We also assume basic knowledge of monoidal categories. Every monoidal category is monoidally equivalent to a strict monoidal category [41, Section XI, Subsection 3, Theorem 1]. Therefore we are safe to suppress all unitors and associators, treating them as identities. All our monoidal structures are additive and symmetric, and we denote the tensor-product functor by $\otimes$ and the tensor unit by 1. Where a monoidal category is closed, we denote the internal hom-functor by hom.

Basic knowledge about triangulated categories is also assumed. Here, unless stated otherwise, we denote the shift functor by Σ. We refer to distinguished triangles and exact triangles interchangeably. We say that a functor $F : \mathcal{K} \to \mathcal{T}$ between triangulated categories $\mathcal{K}$ and $\mathcal{T}$ is a **triangulated functor** or is **exact** if F maps exact triangles to exact triangles. The natural isomorphism $F \circ \Sigma \cong \Sigma \circ F$ will usually remain implicit. Let us introduce the following definition.

Definition 2.1.1. Let $\mathscr{C}$ be a category and $f : X \to Y$ be a morphism in $\mathscr{C}$. A morphism $f' : Y \to Z$ is said to be a **weak cokernel** or **pseudocokernel** of f

if $f' \circ f = 0$ and given any morphism $g : Y \to Y'$ which satisfies $g \circ f = 0$, there exists some $g' : Z \to Y'$ such that $g = g' \circ f'$.

A morphism $f'' : W \to X$ is said to be a **weak kernel** or **pseudokernel** of f if $f \circ f'' = 0$ and given any morphism $h : X' \to X$ which satisfies $f \circ h = 0$, there exists some $h' : X' \to W$ such that $h = f'' \circ h'$.

We frequently use the following property which follows easily from the axioms of a triangulated category.

Lemma 2.1.2. *Suppose $\mathcal{T}$ is a triangulated category and $X \xrightarrow{f} Y \xrightarrow{f'} Z \to \Sigma X$ is an exact triangle in $\mathcal{T}$. Then f is a weak kernel of f' and f' is a weak cokernel of f.*

For $\mathcal{T}$ a rigidly-compactly generated tensor triangulated category, we distinguish between $\mathcal{T}$-tensor-closed definable subcategories, that is definable subcategories which are closed under tensoring with any $X \in \mathcal{T}$ and definable tensor-ideals, which are also triangulated. For a full subcategory $\mathcal{X} \subseteq \mathcal{T}$, we denote by $\langle \mathcal{X} \rangle^{\mathrm{def}^{\otimes}}$ and $\langle \mathcal{X} \rangle^{\mathrm{def}^{\otimes\Delta}}$ the smallest $\mathcal{T}$-tensor-closed definable subcategory containing $\mathcal{X}$ and the smallest definable tensor-ideal containing $\mathcal{X}$ respectively. We also use the notation $\langle I \rangle^{\mathrm{cohom}}$ to denote the smallest cohomological ideal containing I for some subclass $I \subseteq \mathrm{morph}(\mathcal{T}^c)$ where $\mathcal{T}$ is a compactly generated triangulated category. Given a topological space X we denote by $\mathbb{O}(X)$ the frame of open subsets of X.

We also assume some very basic knowledge of the model theory of modules including the use of first order many-sorted languages.

2.2 Day convolution product

Given a small preadditive category, $\mathscr{C}$, with a monoidal structure, we will use Day convolution product to induce a monoidal structure on $\mathscr{C}$-Mod.

Theorem 2.2.1. *[20, Theorem 3.3 and Theorem 3.6] Given a complete and cocomplete closed symmetric monoidal category V, and a small (symmetric) monoidal V-enriched category $\mathscr{C}$, the category of V-enriched functors from $\mathscr{C}$ to V, $V[\mathscr{C}, V]$, is a monoidal category admitting a (symmetric) closed monoidal structure.*

Throughout our V (as above) will be the category of abelian groups, $\mathbf{Ab}$. Given a symmetric monoidal structure $(\otimes, 1)$ on a small preadditive category $\mathcal{A}$, we may refer to the Day convolution product on the functor category $\mathcal{A}\text{-Mod} := (\mathcal{A}, \mathbf{Ab})$ as the 'induced monoidal structure' or 'induced tensor product' and denote the tensor product functor by $\otimes$. By Theorem 2.2.1, the induced monoidal structure on $\mathcal{A}$-Mod is closed, that is, for every $X \in \mathcal{A}\text{-Mod}$, $X \otimes - : \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}$ has a right adjoint functor which we will denote by $\mathrm{hom}(X, -) : \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}$ and call the **internal hom-functor**.

Since for each $F \in \mathcal{A}$-Mod, $F \otimes -$ is a left adjoint, it is right exact and commutes with direct limits. Furthermore, by definition of Day convolution product, given representable functors $(A, -)$ and $(B, -)$ in $\mathcal{A}$-Mod, we have $(A, -) \otimes (B, -) \cong (A \otimes B, -)$. Thus, by right exactness, if $F \in \mathcal{A}$-mod has presentation $(B, -) \xrightarrow{(f,-)} (A, -) \to F \to 0$, with $f : A \to B$ a morphism in $\mathcal{A}$, then $(C, -) \otimes F$ has presentation $(C \otimes B, -) \xrightarrow{(C \otimes f, -)} (C \otimes A, -) \to (C, -) \otimes F \to 0$.

Notation 2.2.2. Given an additive (skeletally) small category $\mathcal{A}$ every finitely presented module $F \in \mathcal{A}$-mod has a presentation of the form $(B, -) \xrightarrow{(f,-)} (A, -) \xrightarrow{\pi_f} F \to 0$, with $f : A \to B$ in $\mathcal{A}$. We will denote such a functor by F_f.

In the above notation we have $(C, -) \otimes F_f = F_{C \otimes f}$. More generally, $F_f \otimes F_g$ fits into the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
(B \otimes V, -) \arrow[r, "(f \otimes V, -)"] \arrow[d, "(B \otimes g, -)"'] & (A \otimes V, -) \arrow[r] \arrow[d, "(A \otimes g, -)"] & F_f \otimes (V, -) \arrow[r] \arrow[d, "F_f \otimes (g, -)"] & 0 \\
(B \otimes U, -) \arrow[r, "(f \otimes U, -)"] \arrow[d] & (A \otimes U, -) \arrow[r] \arrow[d] & F_f \otimes (U, -) \arrow[r] \arrow[d] & 0 \\
(B, -) \otimes F_g \arrow[r, "(f, -) \otimes F_g"] \arrow[d] & (A, -) \otimes F_g \arrow[r] \arrow[d] & F_f \otimes F_g \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Therefore $F_f \otimes F_g = F_{(f\otimes U, A\otimes g)}$, where $f : A \to B$ and $g : U \to V$ and $(f \otimes U, A \otimes g) : A \otimes U \to (B \otimes U) \oplus (A \otimes V)$ is the canonical map.

Thus, Day convolution product restricts to a monoidal structure on the category of finitely presented additive functors $\mathcal{A}$-mod, which we may also refer to as the 'induced monoidal structure' or 'induced tensor product'. This is exactly the tensor product given in [50, Section 13.3] with $\mathcal{A} = R$-mod. Here we avoid the notation (R-mod)-mod in favour of $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

2.3 Rigid monoidal categories

In this section we will outline the definition of a rigid monoidal category.

Definition 2.3.1. Let $\mathscr{C}$ be a symmetric monoidal category. $C^\vee \in \mathscr{C}$ is **dual** to $C \in \mathscr{C}$ if there exists morphisms $\eta_C : 1 \to C^\vee \otimes C$ and $\epsilon_C : C \otimes C^\vee \to 1$ such that $(C^\vee \otimes \epsilon_C) \circ (\eta_C \otimes C^\vee) = \mathrm{id}_{C^\vee}$ and $(\epsilon_C \otimes C) \circ (C \otimes \eta_C) = \mathrm{id}_C$.

An object C in a closed symmetric monoidal category $\mathscr{C}$ is said to be **rigid** if it has a dual. The category $\mathscr{C}$ is said to be **rigid** if every object of $\mathscr{C}$ is rigid.

The following are important consequences of the existence of dual objects.

Proposition 2.3.2. *(e.g. [24, Proposition 1.10.9]) Let $\mathscr{C}$ be a symmetric monoidal category and suppose $C \in \mathscr{C}$ is rigid. Then $C^\vee \otimes -$ is both left and right adjoint to $C \otimes -$.*

Corollary 2.3.3. *Let $\mathscr{C}$ be a closed symmetric monoidal category and suppose $C^\vee$ is dual to C in $\mathscr{C}$. There exists a natural isomorphism* $\hom(C, -) \cong C^\vee \otimes -$.

Corollary 2.3.4. *Let $\mathcal{A}$ be an abelian category with a closed symmetric monoidal structure and suppose $C \in \mathcal{A}$ is rigid. Then $C \otimes - : \mathcal{A} \to \mathcal{A}$ is exact.*

Definition 2.3.5. Let $\mathscr{C}$ be a rigid symmetric monoidal category. Given any morphism, $f : A \to B$ in $\mathscr{C}$, there exists a dual morphism, $f^\vee : B^\vee \to A^\vee$ in $\mathscr{C}$, given by the composition

$$B^\vee \xrightarrow{\eta_A \otimes B^\vee} A^\vee \otimes A \otimes B^\vee \xrightarrow{A^\vee \otimes f \otimes B^\vee} A^\vee \otimes B \otimes B^\vee \xrightarrow{A^\vee \otimes \epsilon_B} A^\vee.$$

2.4 Purity in finitely accessible categories

The results in this section will be stated without proof and we direct the reader to [48], [49] and [51] for more details. Throughout the section, we use [48] and [49] as convenient secondary sources.

Let us recall the definition of a finitely accessible category.

Definition 2.4.1. A category $\mathcal{C}$ is said to be **finitely accessible**, if it has direct limits and there exists a set, $\mathscr{G}$, of finitely presentable objects of $\mathcal{C}$ such that for every $X \in \mathcal{C}$, we can write X as a direct limit of copies of objects of $\mathscr{G}$. That is, $X = \varinjlim_{i \in I} X_i$ where I is some directed indexing set and each $X_i \in \mathscr{G}$. Note that in this case, the full subcategory of finitely presentable objects of $\mathcal{C}$, denoted by $\mathcal{C}^{\mathrm{fp}}$, is skeletally small and we can take $\mathscr{G}$ to consist of a representative of each isomorphism class of $\mathcal{C}^{\mathrm{fp}}$. For the purposes of this thesis we will take 'finitely accessible' to mean additive and finitely accessible.

A category $\mathcal{C}$ is **locally finitely presented** if it is finitely accessible, complete and cocomplete.

Example 2.4.2. *The category $\mathcal{A}$-*Mod *for any ring or skeletally small preadditive category $\mathcal{A}$ is a locally finitely presented category. The skeletally small subcategory of finitely presentable objects is the subcategory of finitely presented modules,* $\mathcal{A}$-mod*.*

Next we define the language for modules over a small preadditive category $\mathcal{A}$.

Definition 2.4.3. Given a (skeletally) small preadditive category $\mathcal{A}$, we define the **language for right** (respectively **left**) **$\mathcal{A}$-modules**, to be the multi-sorted language with a sort for each (isomorphism class of) object of $\mathcal{A}$, a constant symbol 0_A and a binary function symbol $+_A$ of sort A for each object $A \in \mathcal{A}$ and a unary function symbol, f of sort $B \to A$ (respectively $A \to B$), for every morphism $f : A \to B$ in $\mathcal{A}$. We denote this language by $\mathscr{L}_{\mathcal{A}}$ (respectively ${}_{\mathcal{A}}\mathscr{L}$).

Notation 2.4.4. When writing formulas in a many-sorted language $\mathscr{L}$, we use subscripts to indicate the sort of any variable, so x_A is a variable of sort A.

A right (respectively left) $\mathcal{A}$-module, $F : (\mathcal{A})^{\mathrm{op}} \to \mathbf{Ab}$ (respectively $F : \mathcal{A} \to \mathbf{Ab}$), becomes a structure in the language for right (respectively left) $\mathcal{A}$-modules, where the universe is the multi-sorted set $(F(A))_{A\in\mathcal{A}}$, $+_A$ and 0_A give the abelian group structure of $F(A)$ and for each $f : A \to B$ in $\mathcal{A}$, the interpretation of the function symbol $f : B \to A$ (respectively $f : A \to B$) is given by $F(f)$.

A **pp formula** in any language has the form

$$\exists y_1, ..., y_l \bigwedge_{j=1}^{m} \theta_j(x_1, ..., x_n, y_1, ..., y_l)$$

where the θ_js are atomic formulas in the language. Given pp formulas ϕ and ψ in the language $\mathscr{L}_{\mathcal{A}}$ (respectively ${}_{\mathcal{A}}\mathscr{L}$) with the same number of free variables, we say that $\psi \le \phi$ if for all $F \in \text{Mod-}\mathcal{A}$ (respectively $F \in \mathcal{A}\text{-Mod}$), $\psi(F) \subseteq \phi(F)$. $\le$ defines a partial order on the pp formulas and if $\psi \le \phi$ we say that ϕ/ψ is a **pp-pair**. Since $\psi(F)$ and $\phi(F)$ are always additive abelian groups we can form the quotient group $\phi(F)/\psi(F)$. Thus, since morphisms preserve pp formulas, the pp-pair ϕ/ψ gives rise to a functor $\text{Mod-}\mathcal{A} \to \mathbf{Ab}$ (respectively $\mathcal{A}\text{-Mod} \to \mathbf{Ab}$). We say that two pp-pairs ϕ/ψ and ϕ'/ψ' in $\mathscr{L}_{\mathcal{A}}$ (respectively ${}_{\mathcal{A}}\mathscr{L}$) are **equivalent on** Mod-$\mathcal{A}$ (respectively **equivalent on** $\mathcal{A}$-Mod) if for all $F \in \text{Mod-}\mathcal{A}$ (respectively $F \in \mathcal{A}\text{-Mod}$), $\phi(F)/\psi(F) = \phi'(F)/\psi'(F)$. So pp-pairs are equivalent on Mod-$\mathcal{A}$ (respectively $\mathcal{A}$-Mod) if and only if they give rise to the same functor. We can identify a pp formula ϕ with the pp-pair $\phi/\overline{x} = 0$ where $\overline{x}$ matches the free variables of ϕ. Two pp formulas are equivalent if so are their associated pp-pairs.

Let $\mathbb{L}_{\mathcal{A}}^{\mathrm{eq}+}$ (respectively ${}_{\mathcal{A}}\mathbb{L}^{\mathrm{eq}+}$) denote the **category of pp-pairs** in the language for right (respectively left) $\mathcal{A}$-modules. That is, the category with objects given by pp-pairs and morphisms given by pp-definable maps between $\mathcal{A}$-modules (see [48, Section 3.2.2]). Here, a pp formula $\rho(\overline{x}, \overline{y})$ is said to be a pp-definable map from ϕ/ψ to ϕ'/ψ' if for all $F \in \text{Mod-}\mathcal{A}$ (respectively $F \in \mathcal{A}\text{-Mod}$), $\rho(F)$ is the graph of a group homomorphism from $\phi(F)/\psi(F)$ to $\phi'(F)/\psi'(F)$. Notice that equivalent pp-pairs are isomorphic as objects of $\mathbb{L}_{\mathcal{A}}^{\mathrm{eq}+}$ (respectively ${}_{\mathcal{A}}\mathbb{L}^{\mathrm{eq}+}$). We have the following theorem.

Theorem 2.4.5. *[48, Theorem 10.2.30] The category* $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}}$ *is equivalent to the category* $\mathbb{L}_{\mathcal{A}}^{\text{eq}+}$ *of pp-pairs in the language for right* $\mathcal{A}$*-modules. Similarly, the category* $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$ *is equivalent to the category* ${}_{\mathcal{A}}\mathbb{L}^{\text{eq}+}$ *of pp-pairs in the language for left* $\mathcal{A}$*-modules.*

Therefore, Day convolution product induces a tensor product of (equivalence classes of) pp-pairs by asserting that the equivalence in Theorem 2.4.5 is monoidal.

One can define an elementary duality of pp formulas syntactically as follows. A pp formula ϕ in the language $\mathscr{L}_{\mathcal{A}}$ will have the form

$$\exists y_{B_1}, ..., y_{B_m} \bigwedge_{j=1}^{l} \sum_{i=1}^{n} f_{ij}(x_{A_i}) + \sum_{k=1}^{m} g_{kj}(y_{B_k}) = 0_{C_j},$$

where $f_{ij} : C_j \to A_i$ and $g_{kj} : C_j \to B_k$ are morphisms in $\mathcal{A}$ which give rise to unary function symbols of the opposite arity. In the language for right $\mathcal{A}$-modules, it is convention for the unary function symbols to act on the right. Thus we rewrite ϕ as

$$\exists y_{B_1}, ..., y_{B_m} \bigwedge_{j=1}^{l} \sum_{i=1}^{n} x_{A_i} f_{ij} + \sum_{k=1}^{m} y_{B_k} g_{kj} = 0_{C_j}$$

or for short

$$\exists \overline{y}\ (\overline{x}, \overline{y}) \begin{pmatrix} F \\ G \end{pmatrix} = \overline{0},$$

where F is the $n \times l$ matrix with (i, j)th entry f_{ij} and G is the $m \times l$ matrix with (k, j)th entry g_{kj}. The elementary dual of ϕ is the pp formula $D\phi$ in the language ${}_{\mathcal{A}}\mathscr{L}$ given by

$$\exists \overline{z}\ \begin{pmatrix} I_n & F \\ 0_{m\times n} & G \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = \overline{0},$$

or more specifically

$$\exists z_{C_1}, ..., z_{C_l}\ ((\bigwedge_{i=1}^{n} x_{A_i} + \sum_{j=1}^{l} f_{ij}(z_{C_j}) = 0_{A_i}) \wedge (\bigwedge_{k=1}^{m} g_{kj}(z_{C_j}) = 0_{B_k})).$$

Given a pp formula ψ in ${}_{\mathcal{A}}\mathscr{L}$ of the form

$$\exists \overline{y}\ \begin{pmatrix} F & G \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \overline{0},$$

we define the elementary dual pp formula $D\psi$, in the language $\mathscr{L}_{\mathcal{A}}$ to be

$$\exists \overline{z}\ (\overline{x}, \overline{z}) \begin{pmatrix} I & 0 \\ F & G \end{pmatrix} = \overline{0}.$$

It is straight forward to check that ϕ and $DD\phi$ are equivalent on Mod-$\mathcal{A}$ (in particular isomorphic as objects of $\mathbb{L}_{\mathcal{A}}^{\mathrm{eq}+}$), ψ and $DD\psi$ are equivalent on $\mathcal{A}$-Mod (in particular isomorphic as objects of ${}_{\mathcal{A}}\mathbb{L}^{\mathrm{eq}+}$) and for pp formulas ψ and ϕ for both left and right $\mathcal{A}$-modules, $\psi \leq \phi$ if and only if $D\phi \leq D\psi$, that is D maps pp-pairs to pp-pairs. Furthermore we have the following.

Theorem 2.4.6. *[29, Theorem 2.9] [48, Theorem 3.2.12] Elementary duality induces a duality between the categories $\mathbb{L}_{\mathcal{A}}^{\mathrm{eq}+}$ and ${}_{\mathcal{A}}\mathbb{L}^{\mathrm{eq}+}$.*

Recall from Theorem 2.4.5 that we have equivalences $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}} \simeq \mathbb{L}_{\mathcal{A}}^{\mathrm{eq}+}$ and $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \simeq {}_{\mathcal{A}}\mathbb{L}^{\mathrm{eq}+}$. Therefore elementary duality of pp formulas induces a duality on the functor categories $(\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ and $(\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

Let $\mathcal{A}$ be a skeletally small preadditive category. We define the tensor product of $\mathcal{A}$-modules, a generalisation of tensor product over a ring.

Definition 2.4.7. (see for example [49, Section 3]) The **tensor product of $\mathcal{A}$-modules** is given by a functor $- \otimes_{\mathcal{A}} - : \text{Mod-}\mathcal{A} \times \mathcal{A}\text{-Mod} \to \mathbf{Ab}$ determined on objects (up to isomorphism) by the following two assertions. For every $M \in \text{Mod-}\mathcal{A}$,

(i) $M \otimes_{\mathcal{A}} (A, -) \cong M(A)$ for every $A \in \mathcal{A}$,

(ii) $M \otimes_{\mathcal{A}} -$ is right exact.

The functor is defined on morphisms in the obvious way.

We can now define elementary duality of the functor categories as follows.

Theorem 2.4.8. *(see [49, Theorem 4.5]) Elementary duality induces a duality of categories* $(-)^d : ((\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}})^{\text{op}} \to (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$ *given on objects by mapping* $F = F_f : \text{mod-}\mathcal{A} \to \mathbf{Ab}$, *where* $f : A \to B$ *in* mod-$\mathcal{A}$, *to* $F^d : \mathcal{A}\text{-mod} \to \mathbf{Ab}$ *where* $F^d(L) = (F, - \otimes_{\mathcal{A}} L)$ *for every left* $\mathcal{A}$*-module* L*. In particular,* F^d *has copresentation*

$$0 \to F^d \to A \otimes_{\mathcal{A}} - \xrightarrow{f \otimes_{\mathcal{A}} -} B \otimes_{\mathcal{A}} -.$$

Next we introduce definable subcategories.

Definition 2.4.9. Let $\mathcal{A}$ be a skeletally small preadditive category. A full subcategory $\mathcal{D} \subseteq \text{Mod-}\mathcal{A}$ (respectively $\mathcal{D} \subseteq \mathcal{A}\text{-Mod}$) is said to be a **definable subcategory** if it has form

$$\mathcal{D} = \{X : \phi_\lambda(X)/\psi_\lambda(X) = 0 \ \forall \lambda \in \Lambda\}$$

where $\{\phi_\lambda/\psi_\lambda\}_{\lambda \in \Lambda}$ is a set of pp-pairs in the language $\mathscr{L}_{\mathcal{A}}$ (respectively ${}_{\mathcal{A}}\mathscr{L}$).

Notation 2.4.10. Let $\mathcal{C}$ be a finitely accessible category. For $F \in \mathcal{C}^{\text{fp}}\text{-mod}$, denote by $\overrightarrow{F} : \mathcal{C} \to \mathbf{Ab}$ the unique extension of F to $\mathcal{C}$ which commutes with direct limits ([7, page 4–5], also see [48, Proposition 10.2.41]).

By the equivalences in Theorem 2.4.5, $\mathcal{D} \subseteq \text{Mod-}\mathcal{A}$ is definable if and only if there is a collection of finitely presented functors $\mathsf{S} \subseteq (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}}$ such that $X \in \mathcal{D}$ if and only if $\overrightarrow{F}(X) = 0$ for all $F \in \mathsf{S}$.

Definition 2.4.11. Let $\mathscr{A}$ be an abelian category. A full subcategory $\mathsf{S} \subseteq \mathscr{A}$ is said to be a **Serre subcategory**, if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, $A,\ C \in \mathsf{S}$ if and only if $B \in \mathsf{S}$.

If $\mathcal{D} \subseteq \text{Mod-}\mathcal{A}$ is definable then the set $\mathsf{S} = \{F \in (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}} : \overrightarrow{F}(X) = 0,\ \forall X \in \mathcal{D}\}$ is a Serre subcategory (and similarly for the left $\mathcal{A}$-module case). Indeed, we have the following result.

Theorem 2.4.12. *[49, Theorem 8.1] Let* $\mathcal{A}$ *be a skeletally small preadditive category. Then there exist bijections between:*

(i) the Serre subcategories $\mathsf{S} \subseteq (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\text{fp}}$,

(ii) the Serre subcategories $\mathsf{S} \subseteq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$*,*

(iii) the definable subcategories $\mathcal{D} \subseteq \text{Mod-}\mathcal{A}$ *and*

(iv) the definable subcategories $\mathcal{D} \subseteq \mathcal{A}\text{-Mod}$.

Here the bijection between (i) and (ii) is due to elementary duality, and the bijection between (i) and (iii) is given by $\mathcal{D} \mapsto \{F \in (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}} : \overrightarrow{F}(X) = 0,\ \forall X \in \mathcal{D}\}$ *and* $\mathsf{S} \mapsto \{X \in \text{Mod-}\mathcal{A} : \overrightarrow{F}(X) = 0,\ \forall F \in \mathsf{S}\}$*. The bijection between (ii) and (iv) is the same as the bijection between (i) and (iii) with* mod-$\mathcal{A}$ *and* Mod-$\mathcal{A}$ *replaced by* $\mathcal{A}$-mod *and* $\mathcal{A}$-Mod *respectively.*

Notation 2.4.13. Given a Serre subcategory $\mathsf{S} \subseteq (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ we will denote the elementary dual Serre subcategory by S^d, that is $\mathsf{S}^d = \{F^d : F \in \mathsf{S}\} \subseteq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

Similarly, given a definable subcategory $\mathcal{D} \subseteq \text{Mod-}\mathcal{A}$ we will denote the elementary dual definable subcategory, associated to S^d by annihilation, by $\mathcal{D}^d \subseteq \mathcal{A}\text{-Mod}$.

We will also use this $(-)^d$ notation for the inverse map. That is, if $\mathsf{S} \subseteq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ is a Serre subcategory $\mathsf{S}^d \subseteq (\text{mod-}\mathcal{A}, \mathbf{Ab})^{\mathrm{fp}}$ is the dual Serre subcategory and similarly for definable subcategories.

In order to give an important example of elementary duality, we introduce the following definitions.

Definition 2.4.14. Let $\mathcal{C}$ be a finitely accessible category. A monomorphism $m : X \to Y$ in $\mathcal{C}$ is said to be a **pure monomorphism** if for every $f : A \to B$ in $\mathcal{C}^{\mathrm{fp}}$ and for all morphisms $h : A \to X$ and $h' : B \to Y$ such that $h' \circ f = m \circ h$ there exist some $k : B \to X$ such that $k \circ f = h$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "h'"] \arrow[dl, dashed, "k"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

Remark 2.4.15. If $\mathcal{C}$ is locally finitely presented (that is finitely accessible, complete and cocomplete), pure monomorphisms can be characterised as those monomorphism $m : X \to Y$ which fit into an exact sequence

$$0 \to X \xrightarrow{m} Y \xrightarrow{p} Z \to 0$$

such that for every $A \in \mathcal{C}^{\mathrm{fp}}$,

$$0 \to (A, X) \xrightarrow{(A,m)} (A, Y) \xrightarrow{(A,p)} (A, Z) \to 0$$

is exact in **Ab** (see [49, Theorem 5.2]).

Definition 2.4.16. Let $\mathcal{A}$ be a skeletally small preadditive category. A right $\mathcal{A}$-module, M, is said to be **flat** if $M \otimes_{\mathcal{A}} - : \mathcal{A}\text{-Mod} \to \mathbf{Ab}$ is exact.

Let $\mathcal{C}$ be a finitely accessible category. An object $M \in \mathcal{C}$ is said to be **absolutely pure** if every monomorphism $M \to N$ with domain M is a pure monomorphism. An object $M \in \mathcal{C}$ is said to be **fp-injective** if for every monomorphism $f : A \to B$ with finitely presented cokernel, and every morphism $k : A \to M$, there exists a morphism $l : B \to M$ such that $k = l \circ f$.

Proposition 2.4.17. *[49, Proposition 5.6] Let $\mathcal{C}$ be a locally finitely presented abelian category. Then an object $M \in \mathcal{C}$ is absolutely pure if and only if it is fp-injective.*

Definition 2.4.18. Let $\mathcal{C}$ be a finitely accessible category. An object $A \in \mathcal{C}^{\mathrm{fp}}$ is said to be **finitely generated** if $(A, -)$ preserves direct limits of monomorphisms. An object $A \in \mathcal{C}^{\mathrm{fp}}$ is said to be **coherent** if every finitely generated subobject of A is finitely presented. We say that the category $\mathcal{C}$ is **locally coherent** if the full subcategory of coherent objects, denoted $\mathcal{C}^{\mathrm{coh}}$, is skeletally small and every object $X \in \mathcal{C}$ can be written as a direct limit of a directed system of objects from $\mathcal{C}^{\mathrm{coh}}$.

Example 2.4.19. *[49, Example 8.2] Suppose* $\mathcal{A}$-Mod *is locally coherent (for example $\mathcal{A} = \mathcal{C}^{\mathrm{fp}}$ where $\mathcal{C}$ is a finitely accessible category with products (see [49, Theorem 6.1])). Let* Flat-$\mathcal{A} \subseteq$ Mod-$\mathcal{A}$ *denote the full subcategory of flat right*

$\mathcal{A}$-modules. Similarly, let $\mathcal{A}$-Abs $\subseteq$ $\mathcal{A}$-Mod *denote the full subcategory of absolutely pure left $\mathcal{A}$-modules. Then* Flat-$\mathcal{A}$ *and* $\mathcal{A}$-Abs *are elementary dual definable subcategories.*

Now let $\mathcal{C}$ be a finitely accessible category with products. We define the **canonical language for** $\mathcal{C}$, denoted $\mathscr{L}(\mathcal{C})$, to be the language of right $\mathcal{C}^{\mathrm{fp}}$-modules. We identify objects of $\mathcal{C}$ with structures of the language via the restricted Yoneda functor. That is, given $X \in \mathcal{C}$, we can define a structure in the canonical language for $\mathcal{C}$, with universe $\big(\mathcal{C}(C,X)\big)_{C\in\mathcal{C}^{\mathrm{fp}}}$, the $+$ and 0 in each sort giving the abelian group structure on the hom-set and for each $f : A \to B$ in $\mathcal{C}^{\mathrm{fp}}$, the interpretation in X of the function symbol f of arity $B \to A$ given by $- \circ f = (f,X) : (B,X) \to (A,X)$. With this interpretation in mind, rather than writing the term $f(x_B)$ where f is the function symbol of arity $B \to A$, we will write $x_B \circ f$ or just $x_B f$.

The languages $\mathscr{L}(\mathcal{C})$ and $\mathscr{L}_{\mathcal{C}^{\mathrm{fp}}}$ are the same and therefore, the pp formulas and pp-pairs are the same. However we are most interested in the $\mathscr{L}(\mathcal{C})$-structures induced by the objects $X \in \mathcal{C}$, that is the structures corresponding to the representable functors $(-,X)|_{\mathcal{C}^{\mathrm{fp}}}$. Recall that pp-pairs ϕ/ψ and ϕ'/ψ' in the language $\mathscr{L}_{\mathcal{C}^{\mathrm{fp}}}$ are equivalent on Mod-$\mathcal{C}^{\mathrm{fp}}$ if $\phi(F)/\psi(F) = \phi'(F)/\psi'(F)$ for all functors $F \in$ Mod-$\mathcal{C}^{\mathrm{fp}}$. For pp-pairs in the language $\mathscr{L}(\mathcal{C})$, we assert that ϕ/ψ and ϕ'/ψ' are **equivalent on** $\mathcal{C}$ if

$$\phi(X)/\psi(X) = \phi'(X)/\psi'(X)$$

for all $X \in \mathcal{C}$. Here the $\mathscr{L}(\mathcal{C})$-structure X is equal to the $\mathscr{L}_{\mathcal{C}^{\mathrm{fp}}}$-structure $(-,X)|_{\mathcal{C}^{\mathrm{fp}}}$. So if two pp-pairs are equivalent on Mod-$\mathcal{C}^{\mathrm{fp}}$ then they are certainly equivalent on $\mathcal{C}$.

Since $\mathcal{C}^{\mathrm{fp}}$ is closed under finite direct products, every pp formula in $\mathscr{L}(\mathcal{C})$ is equivalent on $\mathcal{C}$ to a pp formula with one free variable. More specifically we have the following result.

Proposition 2.4.20. *(see e.g. [49, Section 18]) Let $\mathcal{C}$ be a finitely accessible category with arbitrary products. Every pp formula in the canonical language for $\mathcal{C}$ is equivalent on $\mathcal{C}$ to a pp-formula of the form*

$$\exists y_B\ (x_A f = y_B g)$$

where $f : C \to A$ and $g : C \to B$ are morphisms in $\mathcal{C}^{\mathrm{fp}}$.

We define the category $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ of pp-pairs in the canonical language for $\mathcal{C}$ to have objects given by pp-pairs and morphisms given by pp-definable maps but this time we say that a pp formula $\rho(\overline{x}, \overline{y})$ is said to be a pp-definable map from ϕ/ψ to ϕ'/ψ' if for all $X \in \mathcal{C}$, $\rho(X)$ is the graph of a group homomorphism from $\phi(X)/\psi(X)$ to $\phi'(X)/\psi'(X)$. Notice that if two pp-pairs are equivalent on $\mathcal{C}$, then they are isomorphic as objects of $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ but may not be isomorphic as objects in $\mathbb{L}^{\mathrm{eq}+}_{\mathcal{C}^{\mathrm{fp}}}$.

By Theorem 2.4.5, the category $\mathbb{L}^{\mathrm{eq}+}_{\mathcal{C}^{\mathrm{fp}}}$ of pp-pairs in the language for right $\mathcal{C}^{\mathrm{fp}}$-modules is equivalent to $(\text{mod-}\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$. We will show that the category $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ is equivalent to a Serre localisation of $(\text{mod-}\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$.

Now we give the definition of a definable subcategory of a finitely accessible category with products.

Definition 2.4.21. Let $\mathcal{C}$ be a finitely accessible category with products. A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is said to be **definable** if it is closed in $\mathcal{C}$ under products, direct limits and pure subobjects. A **definable category** is a definable subcategory of some finitely accessible category with products.

Remark 2.4.22. We will see below (see Proposition 2.4.27 and the commentary there after) that every definable category is equivalent to a definable subcategory of a module category, Mod-$\mathcal{A}$, for some small preadditive category $\mathcal{A}$.

We can use the definable subcategories of a finitely accessible category with product to define a topology called the Ziegler spectrum.

Definition 2.4.23. We say that an object $E \in \mathcal{C}$ is **pure-injective** if it is injective over pure monomorphisms, that is for every pure monomorphism $m : X \to Y$ in $\mathcal{C}$ and any morphism $k : X \to E$ there exists some $h : Y \to E$ such that $k = h \circ m$.

In fact, each finitely accessible category with products has, up to isomorphism, a set of indecomposable pure-injective objects (see [60, Corollary 4.2(1)]) and each definable subcategory is generated as such by its indecomposable pure-injectives. They form the underlying set of a topological space called the Ziegler spectrum.

Definition 2.4.24. We define the **Ziegler spectrum** of $\mathcal{C}$, denoted $\mathrm{Zg}(\mathcal{C})$, to have underlying set given by the set of isomorphism classes of indecomposable pure-injectives in $\mathcal{C}$, denoted $\mathrm{pinj}_{\mathcal{C}}$, and closed subsets given by

$$\{[X] \in \mathrm{pinj}_{\mathcal{C}} : X \in \mathcal{D}\}$$

where $[X]$ denotes the isomorphism class of the indecomposable pure-injective X and $\mathcal{D}$ runs through the definable subcategories of $\mathcal{C}$.

Proposition 2.4.25. *([60, Theorem 4.9], also see [49, Theorem 14.1]) Let $\mathcal{C}$ be a finitely accessible category with products. The closed subsets described above define a topology on* $\mathrm{pinj}_{\mathcal{C}}$.

Next we show that the definable subcategories of $\mathcal{C}$ can be defined in the same way as the definable subcategories of a module category.

Proposition 2.4.26. *(see [49, Theorem 19.4]) A full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is a definable subcategory if and only if it has form*

$$\mathcal{D} = \{X : \phi_\lambda(X)/\psi_\lambda(X) = 0 \ \forall \lambda \in \Lambda\}$$

where $\{\phi_\lambda/\psi_\lambda\}_{\lambda\in\Lambda}$ is a set of pp-pairs in the canonical language, $\mathscr{L}(\mathcal{C})$.

Indeed, the following result tells us that a finitely accessible category with products $\mathcal{C}$ is equivalent to a definable subcategory of a module category.

Proposition 2.4.27. *[49, Theorem 3.4(2) and Theorem 6.1(b)(v)] Let $\mathcal{C}$ be a finitely accessible category with products. Then $\mathcal{C} \simeq \mathrm{Flat}\text{-}\mathcal{C}^{\mathrm{fp}}$, and $\mathrm{Flat}\text{-}\mathcal{C}^{\mathrm{fp}} \subseteq \mathrm{Mod}\text{-}\mathcal{C}^{\mathrm{fp}}$ is a definable subcategory, where $\mathrm{Flat}\text{-}\mathcal{C}^{\mathrm{fp}} \subseteq \mathrm{Mod}\text{-}\mathcal{C}^{\mathrm{fp}}$ denotes the full subcategory of flat right $\mathcal{C}^{\mathrm{fp}}$-modules.*

Thus the definable subcategories of $\mathcal{C}$ can be viewed as definable subcategories of $\mathrm{Mod}\text{-}\mathcal{C}^{\mathrm{fp}}$. Let $\mathsf{S}_{\mathrm{Flat}} \subseteq (\mathrm{mod}\text{-}\mathcal{C}^{\mathrm{fp}}, \mathbf{Ab})^{\mathrm{fp}}$ denote the Serre subcategory consisting of all functors F such that $\overrightarrow{F}(X) = 0$ for all $X \in \mathrm{Flat}\text{-}\mathcal{C}^{\mathrm{fp}}$. Therefore if

$\mathcal{D} \subseteq \text{Flat-}\mathcal{C}^{\text{fp}} \subseteq \text{Mod-}\mathcal{C}^{\text{fp}}$ is a definable subcategory, then the associated Serre subcategory $\mathsf{S} \subseteq (\text{mod-}\mathcal{C}^{\text{fp}}, \mathbf{Ab})^{\text{fp}}$ (as in Theorem 2.4.12) must contain S_{Flat} since any $F \in \mathsf{S}_{\text{Flat}}$ annihilates all flat right $\mathcal{C}^{\text{fp}}$-modules and so definitely annihilates all modules in $\mathcal{D}$. Consequently, we have a one-to-one correspondence between the definable subcategories of $\mathcal{C}$ and the Serre subcategories of the localisation $(\text{mod-}\mathcal{C}^{\text{fp}}, \mathbf{Ab})^{\text{fp}}/\mathsf{S}_{\text{Flat}}$. Next we show that the localisation $(\text{mod-}\mathcal{C}^{\text{fp}}, \mathbf{Ab})^{\text{fp}}/\mathsf{S}_{\text{Flat}}$ is equivalent to $\mathcal{C}^{\text{fp}}$-mod.

Lemma 2.4.28. *(see [49, Corollary 3.5]) Let $\mathcal{A}$ be a finitely accessible abelian category. Then $\mathcal{A}$ is locally coherent if and only if $\mathcal{A}^{\text{fp}}$ is abelian.*

Lemma 2.4.29. *[48, Theorem 11.1.44] Let $\mathcal{A}$ be a locally coherent abelian category (for example when $\mathcal{A} = \mathcal{C}^{\text{fp}}\text{-Mod}$ where $\mathcal{C}$ is a finitely accessible category with products (see [49, Theorem 6.1])). Then for any pp-pair ϕ/ψ in the language $\mathscr{L}(\mathcal{A})$ there exists some $A' \in \mathcal{C}^{\text{fp}}$ such that for every absolutely pure object $M \in \mathcal{C}$, $\phi(M)/\psi(M) \cong (A', M)$.*

Remark 2.4.30. We have seen that any pp-pair ϕ/ψ in $\mathscr{L}(\mathcal{C})$ gives rise to a finitely presented functor $F_f \in (\mathcal{C}^{\text{fp}}\text{-mod}, \mathbf{Ab})^{\text{fp}}$. Suppose that F_f has presentation $(B,-) \xrightarrow{(f,-)} (A,-) \to F_f \to 0$ where $f : A \to B$ is a morphism in $\mathcal{C}^{\text{fp}}$-mod. As $\mathcal{C}^{\text{fp}}$-Mod is locally coherent and abelian, $\mathcal{C}^{\text{fp}}$-mod is abelian by Lemma 2.4.28 . We can take the A' from Lemma 2.4.29 to be the kernel of f. Indeed, suppose we have exact sequence $0 \to A' \xrightarrow{\ker(f)} A \xrightarrow{f} B$, then the abelian group homomorphism $(A, M)/(f, M) \to (A', M)$ given by $h + (f, M) \mapsto h \circ \ker(f)$ is one-to-one as $\ker(f)$ is a monomorphism and onto as M is absolutely pure and therefore fp-injective by Proposition 2.4.17, so every morphism $A' \to M$ factors through $\ker(f)$.

Lemma 2.4.31. *There exists an equivalence of categories* $(\text{mod-}\mathcal{C}^{\text{fp}}, \mathbf{Ab})^{\text{fp}}/\mathsf{S}_{\text{Flat}} \simeq \mathcal{C}^{\text{fp}}\text{-mod}$.

Proof. By Theorem 2.4.12 and properties of Serre localisation, elementary duality induces an equivalence

$$(\text{mod-}\mathcal{C}^{\text{fp}}, \mathbf{Ab})^{\text{fp}}/\mathsf{S}_{\text{Flat}} \simeq ((\mathcal{C}^{\text{fp}}\text{-mod}, \mathbf{Ab})^{\text{fp}}/\mathsf{S}^{d}_{\text{Flat}})^{\text{op}}.$$

By Example 2.4.19, $\mathsf{S}^d_{\mathrm{Flat}} = \mathsf{S}_{\mathrm{Abs}} \subseteq (\mathcal{C}^{\mathrm{fp}}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ consists of all the functors which annihilate all absolutely pure left $\mathcal{C}^{\mathrm{fp}}$-modules. Thus two functors F and G are isomorphic in $(\mathcal{C}^{\mathrm{fp}}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Abs}}$ if and only if the restriction of $\overrightarrow{F}$ and $\overrightarrow{G}$ to absolutely pure modules are isomorphic. Therefore by Lemma 2.4.29 and Remark 2.4.30, $F_f \mapsto \ker(f) \in \mathcal{C}^{\mathrm{fp}}\text{-mod}$ and $A \mapsto (A, -) \in (\mathcal{C}^{\mathrm{fp}}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Abs}}$ induce an equivalence $((\mathcal{C}^{\mathrm{fp}}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Abs}})^{\mathrm{op}} \simeq \mathcal{C}^{\mathrm{fp}}\text{-mod}$ as required. $\square$

Theorem 2.4.32. *[49, Theorem 22.1] The category $\mathbb{L}(\mathcal{C})^{\mathrm{eq}+}$ of pp-pairs in the canonical language for $\mathcal{C}$ is equivalent to the category $\mathcal{C}^{\mathrm{fp}}$-mod.*

In Theorem 2.4.33 below we summarise the connections between definable subcategories of $\mathcal{C}$, Serre subcategories of $\mathcal{C}^{\mathrm{fp}}$-mod and closed subsets of the Ziegler spectrum.

Theorem 2.4.33. *(see [49, Theorem 14.2]) Let $\mathcal{C}$ be an additive finitely accessible category with products. There exist natural bijections between:*

(i) the definable subcategories $\mathcal{D}$ of $\mathcal{C}$,

(ii) the closed subsets $\mathscr{C}$ of the Ziegler spectrum $\mathrm{Zg}(\mathcal{C})$,

(iii) the Serre subcategories S of $\mathcal{C}^{\mathrm{fp}}$-mod.

The bijection between (i) and (iii) is given by $\mathcal{D} \mapsto \{F \in \mathcal{C}^{\mathrm{fp}}\text{-mod} : \overrightarrow{F}(X) = 0,\ \forall X \in \mathcal{D}\}$ and $\mathsf{S} \mapsto \{X \in \mathcal{C} : \overrightarrow{F}(X) = 0,\ \forall F \in \mathsf{S}\}$ and the bijection between (i) and (ii) is given by $\mathcal{D} \mapsto \mathcal{D} \cap \mathrm{pinj}_{\mathcal{C}}$ and $\mathscr{C} \mapsto \langle \mathscr{C} \rangle^{\mathrm{def}}$.

Below we give key properties of the 2-category anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$. See [51] for full details.

Definition 2.4.34. Let $\mathbb{DEF}$ denote the 2-category with objects given by definable categories, 1-morphisms given by additive functors which preserve direct products and direct limits and 2-morphisms given by natural transformations.

Let $\mathbb{ABEX}$ denote the 2-category with objects given by skeletally small abelian categories, 1-morphisms given by additive exact functors and 2-morphisms given by natural transformations.

Remark 2.4.35. Recall that the objects in a definable subcategory $\mathcal{D}$ of a finitely accessible category $\mathcal{C}$ correspond to a class of $\mathscr{L}(\mathcal{C})$-structures which are axiomatised by some collection of pp-pairs. The 1-morphisms in $\mathbb{DEF}$ are given by model theoretic interpretation functors. An additive functor $I : \mathcal{D} \to \mathcal{D}'$ which commutes with direct products and direct limits gives rise to a model theoretic interpretation of $I(\mathcal{D})$ in $\mathcal{D}$ (e.g. see [48, Corollary 18.2.19]).

Theorem 2.4.36. *[51, Theorem 2.3] There exists a 2-category anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$ given on objects by $\mathscr{A} \mapsto \mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ and $\mathcal{D} \mapsto \mathrm{fun}(\mathcal{D}) := (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$, where $\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ is the category of exact functors from $\mathscr{A}$ to the category of abelian groups and $(\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ is the category of additive functors from $\mathcal{D}$ to the category of abelian groups which commute with direct products and direct limits.*

On morphisms the equivalence works in both directions by mapping an appropriate functor, say F, to precomposition by F, $- \circ F$, and on 2-morphisms it works in the obvious way.

Theorem 2.4.37. *(see [49, Theorem 12.10], [34, Theorem 7.2] for the case $\mathcal{D} = \mathcal{C}$) Given a definable subcategory $\mathcal{D}$ of a finitely accessible category $\mathcal{C}$ with products, $\mathrm{fun}(\mathcal{D}) := (\mathcal{D}, \mathbf{Ab})^{\Pi\to} \simeq \mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ where $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}\text{-mod}$ is the Serre subcategory corresponding to $\mathcal{D}$ (as in Theorem 2.4.33).*

Given a finitely presented functor $F \in \mathcal{C}^{\mathrm{fp}}\text{-mod}$, the restriction to $\mathcal{D}$ of its extension along direct limits, $(\overrightarrow{F})|_{\mathcal{D}} : \mathcal{D} \to \mathbf{Ab}$, commutes with direct products and direct limits and therefore is an object of $\mathrm{fun}(\mathcal{D})$. Let $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}\text{-mod}$ be the Serre subcategory corresponding to $\mathcal{D}$ and recall that $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ is given by formally inverting the morphisms in $\Sigma_{\mathsf{S}} = \{\alpha \in \mathrm{morph}(\mathcal{C}^{\mathrm{fp}}) : \ker(\alpha),\ \mathrm{coker}(\alpha) \in \mathsf{S}\}$. Since every morphism in Σ_{S} is an isomorphism when evaluated at any $D \in \mathcal{D}$, by the universal property of the localisation, the functor $(\overrightarrow{-})|_{\mathcal{D}} : \mathcal{C}^{\mathrm{fp}}\text{-mod} \to \mathrm{fun}(\mathcal{D})$ factors via the localisation $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$. The equivalence in Theorem 2.4.37 is given by the exact functor $(\overrightarrow{-})|_{\mathcal{D}} : \mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S} \to \mathrm{fun}(\mathcal{D})$ induced by this factorisation.

2.5 Purity in compactly generated triangulated categories

Definition 2.5.1. Let $\mathcal{T}$ be a triangulated category with small coproducts. An object $C \in \mathcal{T}$ is said to be **compact** if the canonical morphism $\coprod_{i\in I}(C, X_i) \to (C, \coprod_{i\in I} X_i)$ is an isomorphism for every set $\{X_i : i \in I\}$ of objects of $\mathcal{T}$.

A triangulated category $\mathcal{T}$ with small coproducts is said to be **compactly generated** if $\mathcal{T}^c$, the full subcategory of compact objects, is skeletally small and for every $X \in \mathcal{T}$, $(\mathcal{T}^c, X) = 0$ implies $X = 0$.

Fix a compactly generated triangulated category $\mathcal{T}$ and denote the full subcategory of compact objects by $\mathcal{T}^c$.

The restricted Yoneda functor $H : \mathcal{T} \to \text{Mod-}\mathcal{T}^c$ maps $X \in \mathcal{T}$ to $H_X := \mathcal{T}(-, X)|_{\mathcal{T}^c}$. Note that H is not a faithful functor and we call the morphisms $f : X \to Y$ in $\mathcal{T}$ such that $H_f : \mathcal{T}(-, X)|_{\mathcal{T}^c} \to \mathcal{T}(-, Y)|_{\mathcal{T}^c}$ is zero, **phantom maps**. If we complete a phantom map f to an exact triangle in $\mathcal{T}$, say

$$X \xrightarrow{f} Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X$$

then f' has the following property. For any $k : A \to Y$ such that $A \in \mathcal{T}^c$, if $f' \circ k = 0$ then $k = 0$. That is, f' acts like a monomorphism but only when precomposing with morphisms with source in $\mathcal{T}^c$. We say that f' is a **pure monomorphism**. Note that $f' : Y \to Z$ is a pure monomorphism if and only if $H_{f'}$ is a monomorphism. Indeed, given an exact triangle

$$Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

with f' a pure monomorphism, for every $C \in \mathcal{T}^c$, $0 \to (C, Y) \xrightarrow{(C,f')} (C, Z) \xrightarrow{(C,f'')} (C, \Sigma X) \to 0$ is exact in **Ab**. Such an exact triangle is called a **pure-exact triangle**.

If every pure-exact triangle

$$Y \xrightarrow{f'} Z \xrightarrow{f''} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

splits, then $Y \in \mathcal{T}$ is called **pure-injective**. We denote the full subcategory of indecomposable pure-injectives of $\mathcal{T}$ by $\mathrm{pinj}_{\mathcal{T}}$. This is a skeletally small category (e.g. see [27, p.3]) and we also use the notation $\mathrm{pinj}_{\mathcal{T}}$ to denote the set of isomorphism classes of objects of $\mathrm{pinj}_{\mathcal{T}}$.

The restricted Yoneda functor, $H : \mathcal{T} \to \text{Mod-}\mathcal{T}^c$ restricts to an identification between the pure-injective objects of $\mathcal{T}$ and the injective objects of $\text{Mod-}\mathcal{T}^c$ (see for example [27, p.3]). Furthermore, if $Y \in \mathcal{T}$ is pure-injective and $X \in \mathcal{T}$ is arbitrary, then H induces an isomorphism $\mathcal{T}(X,Y) \cong \text{Mod-}\mathcal{T}^c(H_X, H_Y)$ (e.g. [12, Remark 2.6]).

Next we define the canonical language for $\mathcal{T}$ and give some results about pp formulas in this language.

Definition 2.5.2. The **canonical language for** $\mathcal{T}$, denoted $\mathscr{L}(\mathcal{T})$, has a sort for each isomorphism class of compact objects, a binary function symbol $+_C$ of arity $C \times C \to C$ and a constant symbol 0_C for each sort $C \in \mathcal{T}^c$ and a unary function symbol f of arity $B \to A$ for each morphism $f : A \to B$ in $\mathcal{T}^c$.

Each $X \in \mathcal{T}$ then becomes a $\mathscr{L}(\mathcal{T})$-structure with (C, X) as the underlying set of arity $C \in \mathcal{T}^c$, the interpretations of $+_C$ and 0_C giving the additive abelian group structure on (C, X) and the interpretation of the unary function symbol $f : B \to A$ associated to the morphism $f : A \to B$ in $\mathcal{T}^c$, given by pre-composition with f, that is $- \circ f = (f, X) : (B, X) \to (A, X)$.

We say that two pp formulas, ψ and ϕ, are **equivalent on** $\mathcal{T}$ if for all $X \in \mathcal{T}$, $\psi(X) = \phi(X)$. The following result means we can restrict to working with 'division formulas'.

Proposition 2.5.3. *([27, Proposition 3.1]) Every pp formula in the language $\mathscr{L}(\mathcal{T})$ is equivalent to a pp formula of the form $\exists y_B \; x_A = y_B f$ for some morphism $f : A \to B$ in $\mathcal{T}^c$.*

Let us denote the pp formula $\exists y_B \; x_A = y_B f$ by ϕ_f. Therefore, to each morphism in $\mathcal{T}^c$ we can associate a pp formula ϕ_f.

Proposition 2.5.4. *Pp formulas ϕ_f and $\phi_{f'}$ where $f : A \to B$ and $f' : A \to B'$ are equivalent if and only if there exist morphisms $k : B \to B'$ and $l : B' \to B$ such that $f = l \circ f'$ and $f' = k \circ f$.*

Proof. Suppose ϕ_f and $\phi_{f'}$ are equivalent. Then $\phi_f(B) = \phi_{f'}(B)$, so $f \in \phi_f(B) = \phi_{f'}(B)$. Therefore, there exists $l : B' \to B$ such that $f = l \circ f'$. Similarly, $f' \in \phi_f(B')$ so there exists some $k : B \to B'$ such that $f' = k \circ f$.

Conversely, if $f = l \circ f'$ and $f' = k \circ f$, then $g : A \to X$ is in $\phi_f(X)$ implies $g = g' \circ f$. But then $g = g' \circ l \circ f'$, so $g \in \phi_{f'}(X)$. Similarly, if $g \in \phi_{f'}(X)$ then $g \in \phi_f(X)$. So ϕ_f and $\phi_{f'}$ are equivalent. $\square$

As in the finitely accessible case, to each pp formula ϕ in $\mathscr{L}(\mathcal{T})$, we can associate a functor $F_\phi : \mathcal{T} \to \mathbf{Ab}$ given on objects by $F_\phi(X) = \phi(X)$.

Definition 2.5.5. A functor $F : \mathcal{T} \to \mathbf{Ab}$ is said to be **coherent** if it is an additive functor for which there exists $A, B \in \mathcal{T}^c$ such that F has presentation

$$\mathcal{T}(B, -) \to \mathcal{T}(A, -) \to F \to 0.$$

Denote by $\mathrm{Coh}(\mathcal{T})$ the category of coherent functors.

Notation 2.5.6. In future, we will suppress the notation $\mathcal{T}(-,-)$ in favour of $(-,-)$. For a morphism $f : A \to B$ in $\mathcal{T}^c$, we denote the coherent functor $F : \mathcal{T} \to \mathbf{Ab}$ with presentation $(B,-) \xrightarrow{(f,-)} (A,-) \to F \to 0$ by F_f. Similarly we denote the finitely presented functor $G : (\mathcal{T}^c)^{\mathrm{op}} \to \mathbf{Ab}$ with presentation $(-,A) \xrightarrow{(-,f)} (-,B) \to G \to 0$ by G_f.

Proposition 2.5.7. *[27, Lemma 4.3] Suppose ϕ is a pp formula in the language $\mathscr{L}(\mathcal{T})$. The functor $F_\phi : \mathcal{T} \to \mathbf{Ab}$ given by $X \mapsto \phi(X)$ is a coherent functor and for any coherent functor F, there exists a pp formula ϕ, such that $F \cong F_\phi$ in* $\mathrm{Coh}(\mathcal{T})$.

Indeed given the division formula ϕ_f where $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$ is an exact triangle in $\mathcal{T}^c$, F_{ϕ_f} has presentation

$$(C,-) \xrightarrow{(g,-)} (B,-) \to F_{\phi_f} \to 0.$$

That is, $F_{\phi_f} = F_g$ (see Notation 2.5.6). Equivalent pp formulas give rise to the same coherent functor.

Note that a coherent functor $F_g \in \mathrm{Coh}(\mathcal{T})$ is isomorphic to the cokernel of $(C,-) \xrightarrow{(g,-)} (B,-)$ and to the cokernel of $(C',-) \xrightarrow{(g',-)} (B',-)$ if and only if there exist morphisms $B \to B'$, $B' \to B$, $C \to C'$ and $C' \to C$ such that the following diagrams commute.

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\ B' \arrow[r, "g'"'] & C' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] & C \\ B' \arrow[r, "g'"'] \arrow[u] & C'. \arrow[u] \end{tikzcd}$$

In this case, there exist morphisms of exact triangles in both directions between $A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to \Sigma A'$. Therefore, $F_{\phi_f} \cong F_{\phi_{f'}}$ if and only if there exist morphisms $k : B \to B'$, $l : B' \to B$, $m : A \to A'$ and $n : A' \to A$ such that the following diagrams commute in $\mathcal{T}^c$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & B \arrow[d, "k"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] & B \\ A' \arrow[r, "f'"'] \arrow[u, "n"] & B'. \arrow[u, "l"'] \end{tikzcd}$$

In this case we will say that the pp formulas ϕ_f and $\phi_{f'}$ are **isomorphic**. These pp formulas may not be equivalent, indeed their free variables may be of different sorts. However, for each $X \in \mathcal{T}$, the solution sets $\phi_f(X)$ and $\phi_{f'}(X)$ are isomorphic as abelian groups and give rise to naturally isomorphic coherent functors.

Next we introduce the definable subcategories of a compactly generated triangulated category.

Definition 2.5.8. A full subcategory $\mathcal{D} \subseteq \mathcal{T}$ is said to be **definable** if it has form

$$\mathcal{D} = \{X \in \mathcal{T} : F_i(X) = 0 \ \forall i \in I\},$$

where $\{F_i : i \in I\}$ is a family of coherent functors.

By Proposition 2.5.7 we could also define definable subcategories in terms of pp formulas. Indeed, for every definable subcategory $\mathcal{D}$, there exists a collection of pp formulas, $\{\phi_\lambda : \lambda \in \Lambda\}$ such that $X \in \mathcal{D}$ if and only if the sentences $\forall \overline{x}_\lambda(\phi_\lambda(\overline{x}_\lambda) \leftrightarrow \overline{x}_\lambda = 0)$ hold in the $\mathscr{L}(\mathcal{T})$-structure X for all $\lambda \in \Lambda$. Here for simplicity we have simply written $\overline{x}_\lambda$ to denote the free variables in ϕ_λ without specifying sorts.

Next we provide Krause's Fundamental Correspondence. First we need two more definitions.

Definition 2.5.9. We say that an ideal, $\mathcal{J}$, of morphisms in $\mathcal{T}^c$ is a **cohomological ideal**, if there exists a cohomological functor $F : \mathcal{T}^c \to \mathbf{Ab}$ such that $\mathcal{J} = \{f \in \mathrm{morph}(\mathcal{T}^c) : F(f) = 0\}$.

Definition 2.5.10. Let $\mathrm{pinj}_\mathcal{T}$ denote the set of isomorphism classes of indecomposable pure-injectives in $\mathcal{T}$. We define a topology on $\mathrm{pinj}_\mathcal{T}$ called the **Ziegler spectrum** of $\mathcal{T}$. Say that $\mathscr{C} \subseteq \mathrm{pinj}_\mathcal{T}$ is a closed subset of the Ziegler spectrum of $\mathcal{T}$ if and only if $\mathscr{C} = \mathcal{D} \cap \mathrm{pinj}_\mathcal{T}$ for some definable subcategory $\mathcal{D} \subseteq \mathcal{T}$. We denote the Ziegler spectrum by $\mathrm{Zg}_\mathcal{T}$.

Theorem 2.5.11. *[36, Fundamental Correspondence] There is a bijective correspondence between:*

(i) the definable subcategories of $\mathcal{T}$*,*

(ii) the Serre subcategories of $\mathrm{Coh}(\mathcal{T})$ *,*

(iii) the cohomological ideals of $\mathcal{T}^c$*,*

(iv) the closed subsets of $\mathrm{Zg}_\mathcal{T}$*.*

Here a definable subcategory, $\mathcal{D}$, corresponds to the Serre subcategory, S, of coherent functors which annihilate $\mathcal{D}$, the closed subsets of $\mathrm{Zg}_\mathcal{T}$ are given by $\mathcal{D} \cap \mathrm{pinj}_\mathcal{T}$ and the cohomological ideal of $\mathcal{T}^c$ is given by $\mathcal{J} = \{f \in \mathrm{morph}(\mathcal{T}^c) : (f, X) = 0 \ \forall X \in \mathcal{D}\}$. Throughout the rest of this thesis, we will say that $\mathcal{D}$, S, and $\mathcal{J}$ as above *correspond* if they are associated as in Theorem 2.5.11.

In the remainder of this subsection we will relate definability in $\mathcal{T}$ to definability in $\mathrm{Mod}\text{-}\mathcal{T}^c$. Recall that a right $\mathcal{T}^c$-module M is absolutely pure if every monomorphism with domain M is a pure monomorphism and denote the full subcategory of absolutely pure modules by Abs-$\mathcal{T}^c$. In Mod-$\mathcal{T}^c$ the absolutely pure modules satisfy the following.

Proposition 2.5.12. *[6, Proposition 1.8] Any functor $F \in \text{Mod-}\mathcal{T}^c$ is absolutely pure if and only if it is flat.*

In particular, for every $X \in \mathcal{T}$, the image of the restricted Yoneda functor H_X is both absolutely pure and flat. Recall that when Mod-$\mathcal{T}^c$ is locally coherent, Abs-$\mathcal{T}^c$ is definable (see Example 2.4.19). Denote by Zg(Abs-$\mathcal{T}^c$) the intersection $\text{pinj}_{\text{Mod-}\mathcal{T}^c} \cap \text{Abs-}\mathcal{T}^c$ with the subspace topology.

The following Theorem was proved in [6].

Theorem 2.5.13. *[6, Theorem 1.10] The restricted Yoneda functor induces a homeomorphism between $\text{Zg}(\mathcal{T})$ and $\text{Zg}(\text{Abs-}\mathcal{T}^c)$.*

Corollary 2.5.14. *The restricted Yoneda functor $H : \mathcal{T} \to \text{Mod-}\mathcal{T}^c$ induces a bijective correspondence between the definable subcategories of $\mathcal{T}$ and the definable subcategories of Abs-$\mathcal{T}^c$, the absolutely pure right $\mathcal{T}^c$-modules.*

2.6 Rigidly-compactly generated tensor triangulated categories

Definition 2.6.1. A triangulated category $\mathcal{K}$ is said to be **tensor triangulated** (or a tt-category) if it has a symmetric monoidal structure $(\otimes, 1)$ such that the tensor product $- \otimes - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is triangulated in each variable.

Definition 2.6.2. If $\mathcal{T}$ is a tensor triangulated category which is compactly generated and the subcategory of compact objects $\mathcal{T}^c$ forms a rigid monoidal subcategory such that $(-)^\vee : (\mathcal{T}^c)^{\text{op}} \to \mathcal{T}^c$ is an exact functor, we say that $\mathcal{T}$ is a **rigidly-compactly generated tensor triangulated category**.

2.6.1 Tensor triangular geometry

We will give some brief background on tensor triangular geometry. Most of the below holds in a more general setting, however here we focus on the Balmer spectrum of a skeletally small rigid monoidal category. See [57] for a short survey giving more details.

Definition 2.6.3. Suppose $\mathcal{K}$ is a skeletally small triangulated category. A subcategory $I \subseteq \mathcal{K}$ is said to be:

(i) **closed under extensions** or **extension-closed** if for every exact triangle $X \to Y \to Z \to \Sigma X$, if $X,\ Z \in I$ then $Y \in I$.

(ii) **shift-closed** if $X \in I$ if and only if $\Sigma X \in I$, equivalently, I is closed under both positive and negative powers of the shift functor Σ.

(iii) **triangulated** if it is both closed under extensions and shift-closed.

(iv) **thick**, if it is triangulated and closed under direct summands.

Definition 2.6.4. Suppose $\mathcal{K}$ is a skeletally small rigid symmetric tensor triangulated category. A thick subcategory $I \subseteq \mathcal{K}$ is said to be a **thick tensor-ideal** if for every $X \in I$ and $Y \in \mathcal{K}$, $X \otimes Y \in I$. We denote the lattice of thick tensor-ideals of $\mathcal{K}$ by $\mathrm{Thick}^{\otimes}(\mathcal{K})$. A thick tensor-ideal I is said to be **radical** if for all $X \in \mathcal{K}$, if $X^{\otimes n} = X \otimes ... \otimes X \in I$ for some $n \geq 1$ then $X \in I$. A thick tensor-ideal $P \subseteq \mathcal{K}$ is said to be **prime** if for all $X, Y \in \mathcal{K}$, if $X \otimes Y \in P$ then $X \in P$ or $Y \in P$. We call these the **prime tensor-ideals** of $\mathcal{K}$.

Remark 2.6.5. In the case that $\mathcal{K}$ is rigid, every tensor-ideal is radical, by for example [57, Remark 1.8].

Note that since $\mathcal{K}$ is skeletally small, and all our subcategories are closed under isomorphism, there is a set of prime tensor-ideals of $\mathcal{K}$. Let $\mathrm{Spc}(\mathcal{K})$ denote the set of prime tensor-ideals of $\mathcal{K}$. We define a topology on $\mathrm{Spc}(\mathcal{K})$ which we call the **Balmer spectrum** of $\mathcal{K}$ (as introduced in [8]). Given any collection of objects $\mathcal{X} \subseteq \mathcal{K}$, we define a closed subset of $\mathrm{Spc}(\mathcal{K})$ to be

$$Z(\mathcal{X}) = \{P \in \mathrm{Spc}(\mathcal{K}) : \mathcal{X} \cap P = \emptyset\}.$$

If $X \in \mathcal{K}$ we call $Z(\{X\})$ the **support** of X and denote it by $\operatorname{supp}(X)$. Note that $Z(\mathcal{X}) = \bigcap_{X \in \mathcal{X}} \operatorname{supp}(X)$ so the supports of objects in $\mathcal{K}$ form a basis of closed sets.

Next we see that the Balmer spectrum satisfies a universal property.

Definition 2.6.6. Given a (small) tensor triangulated category $(\mathcal{K}, \otimes, 1)$ a **support data** on $\mathcal{K}$ is a pair (X, σ) where X is a topological space and σ assigns to each $A \in \mathcal{K}$ a closed subset $\sigma(A)$ of X such that the following conditions hold:

(i) $\sigma(0) = \emptyset$ and $\sigma(1) = X$

(ii) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$

(iii) $\sigma(\Sigma A) = \sigma(A)$

(iv) $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$ for all triangles $A \to B \to C \to \Sigma A$

(v) $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$.

Definition 2.6.7. Given a topological space X, a subset $Y \subseteq X$ is said to be **specialization closed** if $Y = \bigcup_{y \in Y} \overline{\{y\}}$, where $\overline{\{y\}}$ denotes the smallest closed subset of X containing y.

Theorem 2.6.8. *([8, Theorem 3.2 and Theorem 5.2], [18, Proposition 6.1])* $(\operatorname{Spc}(\mathcal{K}), \operatorname{supp}(-))$ *is a support data and for any support data* (X, σ)*, there exists a unique continuous map* $f : X \to \operatorname{Spc}(\mathcal{K})$ *such that* $\sigma(A) = f^{-1}(\operatorname{supp}(A))$ *for any* $A \in \mathcal{K}$*. Moreover, if* X *is a spectral space and there exists a bijection*

$$\theta : \left\{ \begin{matrix} \text{specialization closed} \\ \text{subsets of } X \end{matrix} \right\} \to \left\{ \begin{matrix} \text{radical thick tensor-ideals} \\ \text{of } \mathcal{K} \end{matrix} \right\},$$

given by $Y \mapsto \{A \in \mathcal{K} : \sigma(A) \subseteq Y\}$ *and* $\mathcal{J} \mapsto \sigma(\mathcal{J}) := \cup_{A \in \mathcal{J}} \sigma(A)$*, then* f *is a homeomorphism. In this case we call* (X, σ) *a* ***classifying support data****.*

Definition 2.6.9. Given a spectral topological space X the **Thomason subsets** of X are of the form $\bigcup_{i \in I} Y_i$ where each Y_i is a closed subset of X with quasi-compact open complement.

By [57, Remark 1.29] the Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ have form $\bigcup_{X \in \mathcal{X}} \mathrm{supp}(X)$, where $\mathcal{X}$ is some collection of objects of $\mathcal{K}$. We denote by $\mathrm{Thom}(\mathcal{K})$ the collection of Thomason subsets of $\mathrm{Spc}(\mathcal{K})$. Note that $\mathrm{Thom}(\mathcal{K})$ forms a lattice with join given by union and binary meet given by intersection.

Definition 2.6.10. (see [22, Definition 0.1]) Given a spectral topological space X we can define the **Hochster dual** of X, denoted X^*. This is the topology on the same set of points, X, with the closed sets generated by the quasi-compact open sets in the original topology.

In [30], Hochster showed that X^* is also spectral and $(X^*)^* \simeq X$.

By ([57, Theorem 1.21]), $\mathrm{Spc}(\mathcal{K})$ is spectral for any skeletally small tensor triangulated category $\mathcal{K}$ with a closed monoidal structure. The quasi-compact opens in $\mathrm{Spc}(\mathcal{K})$ are those of the form $U(X) = \mathrm{Spc}(\mathcal{K}) \backslash \mathrm{supp}(X)$ for $X \in \mathcal{K}$, ([8, Proposition 2.14]). Therefore $(\mathrm{Spc}(\mathcal{K}))^*$ has closed sets of the form

$$\bigcap_{X \in \mathcal{X}} (\mathrm{Spc}(\mathcal{K}) \backslash \mathrm{supp}(X)),$$

where $\mathcal{X}$ is some class of objects of $\mathcal{K}$.

Hence the open subsets of the Hochster dual of the Balmer spectrum of $\mathcal{K}$ are those of the form $\bigcup_{X \in \mathcal{X}} \mathrm{supp}(X)$, that is, exactly the Thomason subsets of $\mathrm{Spc}(\mathcal{K})$.

Theorem 2.6.11. *(see [57, Theorem 1.30]) Let $\mathcal{K}$ be a rigid, skeletally small, tensor triangulated category. There exists an isomorphism of lattices*

$$\sigma : \mathrm{Thick}^{\otimes}(\mathcal{K}) \xrightarrow{\sim} \mathrm{Thom}(\mathcal{K}),$$

given by $\sigma : I \mapsto \bigcup_{X \in I} \mathrm{supp}(X)$ *with inverse* $\tau : \mathrm{Thom}(\mathcal{K}) \xrightarrow{\sim} \mathrm{Thick}^{\otimes}(\mathcal{K})$ *given by* $\tau : V \mapsto \{X \in \mathcal{K} : \mathrm{supp}(X) \subseteq V\}$.

2.6.2 Two examples

In this section we will give more details of the tensor triangulated structure in two examples.

2.6.2.1 The derived category of a module category

Suppose R is a commutative ring. Let us describe the rigidly-compactly generated tensor triangulated structure on the unbounded derived category $D(R\text{-Mod})$.

$D(R\text{-Mod})$ is a localisation of the category of chain complexes $\mathrm{Ch}(R\text{-Mod})$ formed by formally inverting all quasi-isomorphisms. We will index a (co)chain complex $X^\bullet$ by

$$X^\bullet : ... \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \to ....$$

The translation functor Σ in $D(R\text{-Mod})$ maps $X^\bullet$ to

$$X^\bullet[1] : ... \to X^{-1} \xrightarrow{-d^{-1}} X^0 \xrightarrow{-d^0} X^1 \xrightarrow{-d^1} X^2 \xrightarrow{-d^2} X^3 \to ...$$

and will be denoted by [1]. That is the ith degree of $X^\bullet[1]$ is X^{i+1}. For an R-module, M, we will denote by $M[-n]$ the chain complex (or corresponding object of $D(R\text{-Mod})$) given by M concentrated in the nth degree and zeros elsewhere.

Definition 2.6.12. Given a chain morphism $f : X^\bullet \to Y^\bullet$ we define the **mapping cone** of f to be the complex

$$\mathrm{cone}(f) : ... \to X^0 \oplus Y^{-1} \xrightarrow{\begin{bmatrix} -d_X^0 & 0 \\ f_0 & d_Y^{-1} \end{bmatrix}} X^1 \oplus Y^0 \xrightarrow{\begin{bmatrix} -d_X^1 & 0 \\ f_1 & d_Y^{0} \end{bmatrix}} X^2 \oplus Y^1 \to ...$$

where the ith degree is $X^{i+1} \oplus Y^i$.

The distinguished triangles in $D(R\text{-Mod})$ are those isomorphic to a triangle of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \to \mathrm{cone}(f) \to X^\bullet[1].$$

The compact objects of $D(R\text{-Mod})$ are given by the perfect complexes $D^{\mathrm{perf}}(R\text{-Mod}) \cong K^b(R\text{-proj})$ that is the complexes isomorphic in $D(R\text{-Mod})$ to bounded complexes of finitely generated projective objects.

The tensor product on $D(R\text{-Mod})$ is given by the left derived tensor product $- \otimes_R^{\mathbf{L}} -$ and the tensor-unit is given by $R[0]$. The internal hom-functor is given by the right derived hom $\mathbf{R}\mathrm{Hom}(-,-)$ and the dualisable objects coincide with the compact objects. With this monoidal structure, $D(R\text{-Mod})$ forms a rigidly-compactly generated tensor triangulated category.

2.6.2.2 The stable module category

Throughout this section, let G be a finite group and k be a field. We denote the group algebra by kG, the category of finitely generated left kG-modules by kG-mod and the category of all kG-modules by kG-Mod.

kG is a cocommutative Hopf algebra with comultiplication given by diagonal action on the elements of G and extending k-linearly, the counit is determined by $g \mapsto 1$ for all $g \in G$ and the antipode takes $g \in G$ to g^{-1}. A consequence of this Hopf algebra structure is the following. Given kG-modules M and N, $M \otimes_k N$ and $\mathrm{Hom}_k(M, N)$ both have kG-module structures where the action of kG on $M \otimes_k N$ is determined by $g(m \otimes_k n) = gm \otimes_k gm$ and the action of kG on $\mathrm{Hom}_k(M, N)$ is determined by $(gf) : M \to N$ satisfying $(gf)(m) = gf(g^{-1}m)$. If M and N are finite dimensional over k (equivalently finitely generated as kG-modules) then $M^{\vee\vee} \cong M$ and $M^{\vee} \otimes_k N \cong \mathrm{Hom}_k(M, N)$ as kG-modules (e.g. [14, Section 3.1]). Therefore, if M is finitely generated we call the kG-module $\mathrm{Hom}_k(M, k)$ the **dual** of M and denote it by $M^{\vee}$. In other words $\otimes_k$ defines a closed symmetric monoidal structure on kG-Mod such that the full subcategory of finitely generated modules kG-mod is a rigid monoidal subcategory.

Now let us describe the tensor triangulated structure of the stable module category. To define the stable module category in full generality we introduce the following definition.

Definition 2.6.13. (e.g. [55, Section 18]) A ring R is quasi-Frobenius (QF) if it is both right and left artinian and right and left self-injective.

Example 2.6.14. *Let G be a finite group and k be a field. The group algebra kG is a quasi-Frobenius ring.*

Definition 2.6.15. Given a QF ring R, the **stable module category** R-$\underline{\mathrm{Mod}}$

has the same objects as R-Mod and morphisms given by

$$\underline{\mathrm{Hom}}(X,Y) = \mathrm{Hom}_R(X,Y)/\mathrm{Proj}(X,Y)$$

where $\mathrm{Proj}(X,Y)$ denotes the group of morphisms from X to Y which factor through a projective module.

Now let us restrict to the case where R is the group algebra of a finite group G over a field k. In this case kG-$\underline{\mathrm{Mod}}$ is a rigidly-compactly generated tensor triangulated category. Indeed, the full subcategory of projective objects is a tensor-ideal and therefore the closed symmetric monoidal structure given by $\otimes_k$, induces a monoidal structure on the stable module category.

The shift functor is $\Sigma = \Omega^{-1}$ where for any $M \in R$-Mod, given an injective envelope $0 \to M \to I$, $\Omega^{-1}(M)$ fits into a short exact sequence $0 \to M \to I \to \Omega^{-1}(M) \to 0$ in R-Mod. Every short exact sequence $0 \to X \to Y \to Z \to 0$ in kG-Mod, gives rise to an exact triangle $X \to Y \to Z \to \Omega^{-1}X$ in kG-$\underline{\mathrm{Mod}}$ and every exact triangle in kG-$\underline{\mathrm{Mod}}$ arises in this way.

The subcategory of compact objects is the full subcategory of finitely generated modules, denoted by kG-$\underline{\mathrm{mod}}$. Note that the dual of a compact object $M \in kG$-$\underline{\mathrm{mod}}$ is given by $M^\vee = \mathrm{Hom}_k(M,k)$.

By Maschke's Theorem if the characteristic of k is coprime to $|G|$, kG is semi-simple. Therefore, for the rest of this section let k denote a field of positive characteristic p and G denote a finite group such that p divides the order of G.

The following theorem characterises the representation type of different group algebras kG.

Theorem 2.6.16. *(see [14, Theorem 4.4.4]) Let G be a finite group and k be an infinite field of characteristic p.*

(i) kG is of finite representation type if and only if G has cyclic Sylow p-subgroups.

(ii) kG is of domestic representation type if and only if $\mathrm{char}\, k = 2$ and the Sylow 2-subgroups of G are isomorphic to the Klein four group.

(iii) kG has tame representation type if and only if $\mathrm{char}\, k = 2$ *and the Sylow 2-subgroups of G are dihedral, semidihedral or generalised quaternion.*

(iv) In all other cases kG is of wild representation type.

Throughout this thesis we will look more closely at the following two examples.

Examples 2.6.17. *(i) (Finite representation type) Suppose $G = \langle g \mid g^5 = 1 \rangle$ is the cyclic group of order five and let k be a field of characteristic* 5.

(ii) (Domestic representation type) Let $G = V_4$ be the Klein four group, that is $V_4 = \langle x, y \mid x^2 = y^2 = [x, y] = e_G \rangle \cong C_2 \times C_2$ and k be an algebraically closed field of characteristic 2.

The Ziegler spectrum of the stable module category of a QF ring is described in [27] as follows. Let StZg_R denote the subset of non-projective elements of $\mathrm{Zg}_{R\text{-}\mathrm{Mod}}$ with the subspace topology. Then the set of indecomposable pure-injectives in $R\text{-}\underline{\mathrm{Mod}}$, denoted by $\mathrm{Zg}(R\text{-}\underline{\mathrm{Mod}})$, can be identified with StZg_R. Indeed, the topology on $\mathrm{Zg}_{R\text{-}\underline{\mathrm{Mod}}}$ corresponds to the subspace topology on StZg_R. Therefore, we have the following.

Proposition 2.6.18. *[27, Proposition 6.1] The Ziegler spectrum of a QF ring R is homeomorphic to the disjoint union*

$$\mathcal{O} \sqcup \mathrm{Zg}_{R\text{-}\underline{\mathrm{Mod}}},$$

where $\mathcal{O}$ denotes the finite clopen subset of indecomposable injectives (equivalently indecomposable projectives).

In particular we have the following two examples.

Example 2.6.19. *Suppose $G = \langle g \mid g^5 = 1 \rangle$ is the cyclic group of order 5 and let k be a field of characteristic* 5 *as in Example 2.6.17 (i). Then $kG \cong k[T]/(T^5)$ under the isomorphism $T \mapsto g - 1$. Set $M_i = k[T]/(T^i)$ for $i = 1, ..., 5$. The M_i form a complete list of the indecomposable finite dimensional modules up to isomorphism, without repetitions (e.g. [14, Section 4.10]). These five modules* *form the points of the Ziegler spectrum of kG-*Mod *and the topology is discrete. The only indecomposable projective modules is M_5 and the Ziegler spectrum of kG-$\underline{\mathrm{Mod}}$ is discrete with four points, M_1, M_2, M_3 and M_4.*

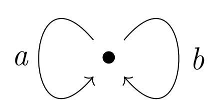
Next let us consider the Ziegler spectrum of the stable module category with G given by the Klein four group and k an alegbraically closed field of characteristic 2. We use the following proposition.

Proposition 2.6.20. *[14, Section 3.14] Let G be a p-group and k be a field of characteristic p. Then kG has a unique minimal left ideal, denoted* $\mathrm{soc}(kG)$ *and the non-projective indecomposable kG-modules correspond to the indecomposable* $kG/\mathrm{soc}(kG)$*-modules.*

Example 2.6.21. *[14, Section 4.3] Let $G = V_4$ be the Klein four group, that is $V_4 = \langle x, y \mid x^2 = y^2 = [x, y] = e_G \rangle \cong C_2 \times C_2$ and k be an algebraically closed field of characteristic 2 as in Example 2.6.17 (ii).*

We consider the algebra $\Lambda = kV_4/\mathrm{soc}(kV_4)$ as in Proposition 2.6.20. Note that $\mathrm{soc}(kV_4) = \langle 1 + x + y + xy \rangle$ *and Λ is* 3*-dimensional, generated by $x - 1$, $y - 1$ and* 1 *with $(x - 1)^2 = (y - 1)^2 = (x - 1)(y - 1) = 0$.*

Let us consider the indecomposable pure-injective kG-modules. First note that $\Lambda = kG/\mathrm{soc}(kG)$ is isomorphic to the path algebra of the quiver Q below factored out by the ideal $I = (a^2, b^2, ab, ba)$.



Thus Λ is a domestic string algebra and the indecomposable pure-injective Λ-modules are string and band modules [53, Theorem 5.1].

For background on string algebras see for example [53], [39] [19], [54]. Here we define string and band modules only for our quiver Q and ideal I.

A string v is a sequence $v = v_n ... v_1$ of arrows from Q (in this case a and b) or inverse arrows from Q (in this case a^{-1} and b^{-1}) such that, aa^{-1}, $a^{-1}a$, bb^{-1} and

$b^{-1}b$ appear nowhere in v, no subsequence of v of the form $v_{i+j}...v_i$ is in I and no subsequence of $v^{-1} = v_1^{-1}...v_n^{-1}$ of the form $v_i^{-1}...v_{i+j}^{-1}$ is in I.

We list the strings and bands below:

(i) *Finite strings*

 (a) $\emptyset$,

 (b) ${}^n(b^{-1}a)$, $(a^{-1}b)^n$, $n \in \mathbb{N}$,

 (c) ${}^n(ab^{-1})$, $(ba^{-1})^n$, $n \in \mathbb{N}$,

 (d) ${}^n(b^{-1}a)b^{-1}$, $b(a^{-1}b)^n$, $n \in \mathbb{Z}^{\geq 0}$,

 (e) ${}^n(ab^{-1})a$, $a^{-1}(ba^{-1})^n$, $n \in \mathbb{Z}^{\geq 0}$.

(ii) *Infinite strings*

 (a) ${}^\infty(b^{-1}a)$, $(a^{-1}b)^\infty$, *(contracting)*

 (b) ${}^\infty(a^{-1}b)$, $(b^{-1}a)^\infty$, *(contracting)*

 (c) ${}^\infty(ba^{-1})$, $(ab^{-1})^\infty$, *(expanding)*

 (d) ${}^\infty(ab^{-1})$, $(ba^{-1})^\infty$, *(expanding)*

(iii) *Bands*

 (a) ba^{-1}, ab^{-1},

Given a finite string $v_n...v_1$, $n \geq 1$ we define the string module $M(v_n...v_1)$ to have underlying vector space k^{n+1} with standard basis $x_0, ..., x_n$ and for $\alpha = a,\ b$ we have

$$\alpha x_i = \begin{cases} x_{i+1} & \text{if } v_{i+1} = \alpha \text{ and } i \neq n \\ x_{i-1} & \text{if } v_i^{-1} = \alpha \text{ and } i \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

This can be visualised by drawing the string as a sequence of arrows where direct arrows are drawn diagonally down from the right to the left and inverse arrows are drawn diagonally down from the left to the right. One then labels the *vertices $x_0, ..., x_n$ from right to left and the action of α on x_i is given by following the arrow α starting at this vertex if it exists and declaring $\alpha x_i = 0$ otherwise.*

For example, the module $M({}^3(b^{-1}a))$ corresponds to the diagram below.

$$\begin{tikzcd}
x_6 \arrow[dr, "b"'] & & x_4 \arrow[dl, "a"] \arrow[dr, "b"'] & & x_2 \arrow[dl, "a"] \arrow[dr, "b"'] & & x_0 \arrow[dl, "a"] \\
& x_5 & & x_3 & & x_1 &
\end{tikzcd}$$

Given a string $v = v_n...v_1$ and its inverse $v^{-1} = v_1^{-1}...v_n^{-1}$, $M(v)$ and $M(v^{-1})$ are isomorphic. Thus, for each (family of) pair(s) of finite strings (a)-(e) given above we get a (family of) finite dimensional indecomposable pure-injective string module(s).

In a similar way, we define infinite string modules. Here we have two choices. Given a left $\mathbb{N}$-string $v = ...v_2v_1$ we define the direct-sum module $M(v)$ to have underlying vector space $\bigoplus_{i\in\mathbb{N}} k$ and the direct-product module $N(v)$ to have underling vector space $\Pi_{i\in\mathbb{N}}k$. In both cases we fix a basis $x_0, x_1, ...$ and the action of $\alpha = a,\ b$ is given by

$$\alpha x_i = \begin{cases} x_{i+1} & \text{if } v_{i+1} = \alpha \\ x_{i-1} & \text{if } v_i^{-1} = \alpha \text{ and } i \neq 0 \\ 0 & \text{otherwise }, \end{cases}$$

applied componentwise in the product module case.

Again an infinite string and its inverse give isomorphic string modules (so we only define $M(v)$ and $N(v)$ for a left $\mathbb{N}$-string here). Infinite string modules can be visualised using infinite diagrams. For example the modules $M({}^\infty(ba^{-1}))$ and $N({}^\infty(ba^{-1}))$ can both be depicted by the diagram below.

$$\begin{tikzcd}
& \cdots & x_3 \arrow[dl, "b"'] \arrow[dr, "a"] & & x_1 \arrow[dl, "b"'] \arrow[dr, "a"] & \\
\cdots & x_4 & & x_2 & & x_0
\end{tikzcd}$$

By [53, Theorem 5.1], $M(v)$ for v a contracting infinite string (cases (a) and (b) above) and $N(v)$ for v an expanding infinite string (cases (c) and (d) above) are indecomposable pure-injective modules (note that so-called "mixed" infinite strings don't occur in this example).

Finally, we define the band modules for this example. For each indecomposable pure-injective $k[T, T^{-1}]$-module $M = (U, \Phi)$, we denote by $B(ab^{-1}, M)$, the Λ-module with underlying vector space $U \oplus U$ and action of a and b given by $a(x, y) = (y, 0)$ and $b(x, y) = (\Phi y, 0)$. Note that since k is algebraically closed, the indecomposable pure-injective $k[T, T^{-1}]$-modules are indexed by $\lambda \in k^\times$, $n \in \mathbb{N} \cup \{-\infty, +\infty\}$, with the additional generic module. We label our band modules accordingly. Here for fixed $\lambda \in k^\times$, $B(ab^{-1}, \lambda, -\infty)$ is the adic module given by the inverse limit $\varprojlim_n B(ab^{-1}, \lambda, n)$ of a coray of epimorphisms and $B(ab^{-1}, \lambda, \infty)$ is the Prüfer module given by the direct limit $\varinjlim_n B(ab^{-1}, \lambda, n)$ of a ray of monomorphisms in a tube in the Auslander-Reiten quiver (see for example [48, Section 8.1.2] for more details).

In particular, the band module $B(ab^{-1}, \lambda, n)$ for $\lambda \in k^\times$ and $n \in \mathbb{N}$ has generators z_1^i and z_2^i for $i = 1, ..., n$ and relations as follows.

$az_1^i = z_2^i$, $i = 1, ..., n$.

$az_2^i = 0$, $i = 1, ..., n$.

$$bz_1^i = \begin{cases} \lambda z_2^1 & i = 1 \\ \lambda z_2^i + z_2^{i-1} & i = 2, ..., n. \end{cases}$$

$bz_2^i = 0$, $i = 1, ..., n$.

Therefore the indecomposable pure-injective Λ-modules are as follows:

(i) *Finite string modules*

(a) $M(\emptyset) \cong k$,

(b) $M({}^n(b^{-1}a))$, $n \in \mathbb{N}$,

(c) $M({}^n(ab^{-1}))$, $n \in \mathbb{N}$,

(d) $M({}^n(b^{-1}a)b^{-1})$, $n \in \mathbb{Z}^{\geq 0}$,

(e) $M({}^n(ab^{-1})a)$, $n \in \mathbb{Z}^{\geq 0}$.

(ii) *Infinite string modules*

  (a) $M({}^{\infty}(b^{-1}a))$,

  (b) $M({}^{\infty}(a^{-1}b))$,

  (c) $N({}^{\infty}(ba^{-1}))$,

  (d) $N({}^{\infty}(ab^{-1}))$,

(iii) *Band modules*

  (a) $B(ab^{-1}, \lambda, n)$, $\lambda \in k^{\times}$, $n \in \mathbb{N}$,

  (b) $B(ab^{-1}, \lambda, -\infty)$, $\lambda \in k^{\times}$ *(adic)*,

  (c) $B(ab^{-1}, \lambda, \infty)$, $\lambda \in k^{\times}$ *(Prüfer)*,

  (d) $B(ab^{-1}, G)$ *where* $G = k(T)$ *is the ring of rational functions as a module (generic).*

It remains to note that the indecomposable pure-injectives of kV_4-$\underline{\mathrm{Mod}}$ *correspond to the indecomposable pure-injectives of* Λ-Mod *by [35, Proposition 1.16] and Proposition 2.6.20.*

Chapter 3

A monoidal analogue of the 2-category anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$

The content of this chapter is from [59].

3.1 The 2-categories $\mathbb{ABEX}^{\otimes}$ and $\mathbb{DEF}^{\otimes}$

In this section we define the 2-categories $\mathbb{ABEX}^{\otimes}$ and $\mathbb{DEF}^{\otimes}$.

Definition 3.1.1. We will say that a functor $F : \mathscr{A} \to \mathscr{B}$ between monoidal categories $(\mathscr{A}, \otimes, 1_{\mathscr{A}})$ and $(\mathscr{B}, \otimes', 1_{\mathscr{B}})$ is **monoidal** if there exists an isomorphism in $\mathscr{B}$, $\epsilon : 1_{\mathscr{B}} \to F(1_{\mathscr{A}})$ and a natural isomorphism $\mu : (\otimes' \circ F \times F) \to (F \circ \otimes)$ satisfying the associativity condition $\mu_{X\otimes Y,Z} \circ (\mu_{X,Y} \otimes' F(Z)) = \mu_{X,Y\otimes Z} \circ (F(X) \otimes' \mu_{Y,Z})$ and the unitality conditions, $\mu_{1_{\mathscr{A}},X} \circ (\epsilon \otimes' F(X)) = \mathrm{id}_{F(X)}$ and $\mu_{X,1_{\mathscr{A}}} \circ (F(X) \otimes' \epsilon) = \mathrm{id}_{F(X)}$.

Definition 3.1.2. Let $\mathbb{ABEX}^{\otimes}$ denote the 2-category with objects given by skeletally small abelian categories equipped with an additive symmetric monoidal structure which is exact in each variable, 1-morphisms given by additive exact monoidal functors and 2-morphisms given by (not necessarily monoidal) natural transformations.

Notation 3.1.3. Given a category $\mathscr{C}$ and morphisms $f : A \to B$ and $k : A \to C$ in $\mathscr{C}$, we will write $f|k$ if there exists some morphism $k' : B \to C$ in $\mathscr{C}$ such that $k = k' \circ f$.

Definition 3.1.4. Let $\mathcal{C}$ be a finitely accessible category with products and an additive symmetric closed monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ is a monoidal subcategory. We say that a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$ is **fp-hom-closed** if for every $A \in \mathcal{C}^{\mathrm{fp}}$ and $X \in \mathcal{D}$, $\mathrm{hom}(A, X) \in \mathcal{D}$, where hom denotes the internal hom-functor.

We say that a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$ satisfies the **exactness criterion** if given morphisms $f : A \to B$ and $g : U \to V$ in $\mathcal{C}^{\mathrm{fp}}$ and a morphism $h : A \otimes U \to X$ in $\mathcal{C}$ where $X \in \mathcal{D}$, if $(f \otimes U)|h$ and $(A \otimes g)|h$ then $(f \otimes g)|h$ (see Notation 3.1.3).

Definition 3.1.5. We define the 2-category $\mathbb{DEF}^{\otimes}$ as follows. Let the objects of $\mathbb{DEF}^{\otimes}$ be given by the triples $(\mathcal{D}, \mathcal{C}, \otimes)$ where $\mathcal{C}$ is a finitely accessible category with products, $\otimes$ is an additive symmetric closed monoidal structure on $\mathcal{C}$ such that $\mathcal{C}^{\mathrm{fp}}$ is a monoidal subcategory and $\mathcal{D}$ is an fp-hom-closed definable subcategory of $\mathcal{C}$ satisfying the exactness criterion. Let the 1-morphisms in $\mathbb{DEF}^{\otimes}$ be given by the additive functors $I : \mathcal{D} \to \mathcal{D}'$ which commute with direct products and direct limits and such that the induced functor $I_0 : \mathrm{fun}(\mathcal{D}') \to \mathrm{fun}(\mathcal{D})$ (given by mapping $F : \mathcal{D}' \to \mathbf{Ab}$ to $F \circ I : \mathcal{D} \to \mathbf{Ab}$ (see [51, Theorem 2.3])) is monoidal. The 2-morphisms are given by (not necessarily monoidal) natural transformations.

Remark 3.1.6. Notice that there exist forgetful 2-functors $\mathscr{F} : \mathbb{ABEX}^{\otimes} \to \mathbb{ABEX}$ and $\mathscr{F} : \mathbb{DEF}^{\otimes} \to \mathbb{DEF}$ which forget the monoidal structure.

3.2 The 2-category anti-equivalence

Theorem 3.2.1. *There exists a 2-category anti-equivalence between* $\mathbb{ABEX}^{\otimes}$ *and* $\mathbb{DEF}^{\otimes}$ *given on objects by* $\mathscr{A} \mapsto (\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}), \mathscr{A}\text{-Mod}, \otimes)$ *where the monoidal structure,* $\otimes$*, on* $\mathscr{A}$-Mod *is induced by the monoidal structure on* $\mathscr{A}$ *via Day convolution product. Conversely, the anti-equivalence maps an object* $(\mathcal{D}, \mathcal{C}, \otimes)$ *in* $\mathbb{DEF}^{\otimes}$ *to the skeletally small abelian category* $\mathrm{fun}(\mathcal{D}) = (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ *with monoidal structure induced by Day convolution product on* $\mathcal{C}^{\mathrm{fp}}$-mod *(see Definition 3.3.9).*

We prove Theorem 3.2.1 in several parts.

3.3 The 2-functor $\theta : (\mathbb{DEF}^{\otimes})^{\mathrm{op}} \to \mathbb{ABEX}^{\otimes}$

First let us define a 2-functor $\theta : (\mathbb{DEF}^{\otimes})^{\mathrm{op}} \to \mathbb{ABEX}^{\otimes}$. On objects, θ maps $(\mathcal{D}, \mathcal{C}, \otimes)$ in $\mathbb{DEF}^{\otimes}$ to $\mathrm{fun}(\mathcal{D})$. In Theorem 3.3.6 we show that the Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$-mod corresponding to $\mathcal{D}$ is a Serre tensor-ideal. We use this in Definition 3.3.9 to define an additive symmetric monoidal structure on $\mathrm{fun}(\mathcal{D})$. We then show that the monoidal structure is exact in each variable in Proposition 3.3.10.

Assumption 3.3.1. Let $\mathcal{C}$ be an additive finitely accessible category with products. Suppose further that $\mathcal{C}$ has an additive closed symmetric monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ is a monoidal subcategory. Induce a monoidal structure on $\mathcal{C}^{\mathrm{fp}}$-Mod via Day convolution product and note that this restricts to a monoidal structure on $\mathcal{C}^{\mathrm{fp}}$-mod. We denote all tensor products by $\otimes$. Note that the monoidal structures on $\mathcal{C}$ and $\mathcal{C}^{\mathrm{fp}}$-Mod are assumed to be closed, and therefore in both cases the tensor product functor, $\otimes$, is right exact in each variable. Furthermore, as $\mathcal{C}$ is an additive finitely accessible category with products, $\mathcal{C}^{\mathrm{fp}}$-Mod is locally coherent [49, Theorem 6.1] and therefore $\mathcal{C}^{\mathrm{fp}}$-mod is an abelian subcategory of $\mathcal{C}^{\mathrm{fp}}$-Mod (e.g. see [48, Theorem E.1.47]). Therefore, every exact sequence in $\mathcal{C}^{\mathrm{fp}}$-mod is exact in $\mathcal{C}^{\mathrm{fp}}$-Mod and consequently the restriction of Day convolution product to $\mathcal{C}^{\mathrm{fp}}$-mod is also right exact in each variable.

Remark 3.3.2. Given a finitely accessible category $\mathcal{C}$ satisfying all the properties in Assumption 3.3.1, we can use the equivalence between the category $\mathbb{L}(\mathcal{C})^{\mathrm{eq+}}$ of pp-pairs in the canonical language for $\mathcal{C}$ and $\mathcal{C}^{\mathrm{fp}}$-mod (see Theorem 2.4.32) to define a monoidal structure on $\mathbb{L}(\mathcal{C})^{\mathrm{eq+}}$. Thus we can define a tensor product of pp-pairs. For example, for $A, B \in \mathcal{C}^{\mathrm{fp}}$, the pp-pairs $(x_A = x_A)/(x_A = 0)$ and $(x_B = x_B)/(x_B = 0)$ correspond to the representable functors $(A, -)$ and $(B, -)$ respectively. Therefore $(x_A = x_A)/(x_A = 0) \otimes (x_B = x_B)/(x_B = 0) = (x_{A\otimes B} = x_{A\otimes B})/(x_{A\otimes B} = 0)$.

Before we prove the correspondence between fp-hom-closed definable subcategories and Serre tensor-ideals (Theorem 3.3.6), we prove some useful lemmas. The first uses the tensor-hom adjunction of a closed monoidal category to find a natural isomorphism between two functors. Notice that if, for every $X \in \mathcal{C}^{\mathrm{fp}}$, the internal hom-functor $\mathrm{hom}(X, -)$, restricts to a functor $\mathrm{hom}(X, -) : \mathcal{C}^{\mathrm{fp}} \to \mathcal{C}^{\mathrm{fp}}$, that is $\mathcal{C}^{\mathrm{fp}}$

forms a closed monoidal subcategory of $\mathcal{C}$, then the statement and proof of the following lemma both simplify.

Lemma 3.3.3. *Let $\mathcal{C}$ be as in Assumption 3.3.1 and induce a monoidal structure on $\mathcal{C}^{\mathrm{fp}}$-Mod via Day convolution product. Then for all $F \in \mathcal{C}^{\mathrm{fp}}$-Mod and $X \in \mathcal{C}^{\mathrm{fp}}$, $(X,-)\otimes F$ is naturally isomorphic to $\overrightarrow{F} \circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}$ where $\mathrm{hom}(X,-) : \mathcal{C} \to \mathcal{C}$ denotes the internal hom-functor and $\mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}} : \mathcal{C}^{\mathrm{fp}} \to \mathcal{C}$ is the restriction to finitely presented objects.*

Proof. First suppose F is finitely presentable with presentation $(B,-) \xrightarrow{(f,-)} (A,-) \xrightarrow{\pi_f} F \to 0$, with $f : A \to B$ in $\mathcal{C}^{\mathrm{fp}}$ and suppose $X \in \mathcal{C}^{\mathrm{fp}}$. Then $(X,-)\otimes F$ has presentation

$$(X \otimes B, -) \xrightarrow{(X\otimes f,-)} (X \otimes A, -) \xrightarrow{\pi_{X\otimes f}} (X,-)\otimes F \to 0,$$

where $X \otimes f : X \otimes A \to X \otimes B$ is in $\mathcal{C}^{\mathrm{fp}}$. For any $Z \in \mathcal{C}^{\mathrm{fp}}$ we have the following diagram in $\mathbf{Ab}$.

$$\begin{tikzcd}
(X\otimes B, Z) \arrow[r, "(X\otimes f,-)_Z"] \arrow[d, "\alpha_B"] & (X\otimes A, Z) \arrow[r, "(\pi_{X\otimes f})_Z"] \arrow[d, "\alpha_A"] & ((X,-)\otimes F)(Z) \arrow[r] \arrow[d, dashed, "\eta_Z"] & 0 \\
(B, \mathrm{hom}(X,Z)) \arrow[r, "(f,\mathrm{hom}(X,-))_Z"] & (A, \mathrm{hom}(X,Z)) \arrow[r, "(\pi_f)_{\mathrm{hom}(X,Z)}"] & (\overrightarrow{F} \circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}})(Z) \arrow[r] & 0
\end{tikzcd}$$

Since the isomorphisms α_B and α_A are natural in A and B respectively, the η_Z form the components of a natural isomorphism $\eta : (X,-)\otimes F \to \overrightarrow{F} \circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}$.

If $F : \mathcal{C}^{\mathrm{fp}} \to \mathbf{Ab}$ is any additive functor then $F = \varinjlim_{i\in I} F_i$ for some finitely presented functors F_i. For each $i \in I$, we have $\eta_i : ((X,-)\otimes F_i) \to \overrightarrow{F_i} \circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}$, defined as above. Furthermore, for any natural transformation $\lambda : F_i \to F_j$ the following diagram commutes, where $\overrightarrow{\lambda} : \overrightarrow{F_i} \to \overrightarrow{F_j}$ denotes the natural transformation with components given by the unique map between direct limits induced by λ.

$$\begin{tikzcd}
(X,-)\otimes F_i \arrow[r, "\eta_i"] \arrow[d, "(X,-)\otimes\lambda"'] & \overrightarrow{F_i}\circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}} \arrow[d, "\overrightarrow{\lambda}_{\mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}}"] \\
(X,-)\otimes F_j \arrow[r, "\eta_j"] & \overrightarrow{F_j}\circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}
\end{tikzcd}$$

Therefore, by the universal property of direct limits, the η_i for $i \in I$ induce a unique natural isomorphism

$$\varinjlim_{i\in I}((X,-)\otimes F_i) \to \varinjlim_{i\in I}(\overrightarrow{F_i}\circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}) = \overrightarrow{F}\circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}.$$

Since $(X,-)\otimes -$ commutes with direct limits,

$$\varinjlim_{i\in I}((X,-)\otimes F_i) \cong (X,-)\otimes \varinjlim_{i\in I}F_i = (X,-)\otimes F.$$

Therefore, we have a natural isomorphism $\eta : (X,-)\otimes F \to \overrightarrow{F}\circ \mathrm{hom}(X,-)|_{\mathcal{C}^{\mathrm{fp}}}$ as required. □

Lemma 3.3.4. *Let $\mathcal{C}$ be as in Assumption 3.3.1. For every $C \in \mathcal{C}^{\mathrm{fp}}$,*

$$\mathrm{hom}(C,-) : \mathcal{C} \to \mathcal{C}$$

commutes with direct limits.

Proof. As $\mathrm{hom}(C,-) : \mathcal{C} \to \mathcal{C}$ is right adjoint to $C \otimes - : \mathcal{C} \to \mathcal{C}$ and by assumption $C\otimes -$ restricts to finitely presented objects, we can apply the proof of [1, Proposition 2.23] to deduce that $\mathrm{hom}(C,-) : \mathcal{C} \to \mathcal{C}$ commutes with direct limits. □

Next, we simplify the criteria for a Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$-mod to be a Serre tensor-ideal.

Lemma 3.3.5. *Let $\mathcal{C}$ be as in Assumption 3.3.1 and suppose S is a Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$-mod. Then S is a Serre tensor-ideal of $\mathcal{C}^{\mathrm{fp}}$-mod if and only if for all*

$C \in \mathcal{C}^{\mathrm{fp}}$ and all $F \in \mathsf{S}$,

$$(C, -) \otimes F \in \mathsf{S}.$$

Proof. ($\Longrightarrow$) Holds by the definition of tensor-ideal.
($\Longleftarrow$) Suppose that for all $C \in \mathcal{C}^{\mathrm{fp}}$ and all $F \in \mathsf{S}$, $(C, -) \otimes F \in \mathsf{S}$. Let $F_f \in \mathsf{S}$ and $F_g \in \mathcal{C}^{\mathrm{fp}}$-mod where F_g has projective resolution

$$(V, -) \xrightarrow{(g,-)} (U, -) \to F_g \to 0,$$

for $g : U \to V$ a morphism in $\mathcal{C}^{\mathrm{fp}}$.

By right exactness of the induced tensor product, we have the exact sequence

$$(V, -) \otimes F_f \to (U, -) \otimes F_f \to F_g \otimes F_f \to 0.$$

By assumption, $(U, -) \otimes F_f$ is an object of S. Therefore $F_g \otimes F_f \in \mathsf{S}$ as S is Serre. Hence S is a tensor-ideal as required. $\square$

Now we are ready to prove the following theorem.

Theorem 3.3.6. *Let $\mathcal{C}$ be an additive finitely accessible category with products. Suppose further that $\mathcal{C}$ has an additive closed symmetric monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ is a monoidal subcategory.*

Let $\mathcal{D}$ be a definable subcategory of $\mathcal{C}$ and $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}$-mod be the corresponding Serre subcategory as in Theorem 2.4.33. Then S is a tensor-ideal of $\mathcal{C}^{\mathrm{fp}}$-mod if and only if $\mathcal{D}$ is fp-hom-closed.

Proof. Recall that the functors in S are exactly those whose unique extension along direct limits annihilates $\mathcal{D}$. Therefore $\mathcal{D}$ is fp-hom-closed if and only if for every $A \in \mathcal{C}^{\mathrm{fp}}$, $X \in \mathcal{D}$ and every $F \in \mathsf{S}$, $\overrightarrow{F}(\hom(A, X)) = 0$. By Lemma 3.3.4 and definition of $\overrightarrow{(-)}$, $\overrightarrow{F} \circ \hom(A, -)$ commutes with direct limits and therefore

$$\overrightarrow{F} \circ \hom(A, -) = \overrightarrow{\overrightarrow{F} \circ \hom(A, -)|_{\mathcal{C}^{\mathrm{fp}}}}.$$

Furthermore, by Lemma 3.3.3, we have

$$(A, -) \otimes F \cong \overrightarrow{F} \circ \hom(A, -)|_{\mathcal{C}^{\mathrm{fp}}} : \mathcal{C}^{\mathrm{fp}} \to \mathbf{Ab},$$

so $\overrightarrow{F} \circ \hom(A,-) \cong \overrightarrow{(A,-)\otimes F}$.

Therefore, $\mathcal{D}$ is fp-hom-closed if and only if for every $A \in \mathcal{C}^{\mathrm{fp}}$, $X \in \mathcal{D}$ and $F \in \mathsf{S}$, $\overrightarrow{(A,-)\otimes F}(X) = 0$, equivalently $(A,-)\otimes F \in \mathsf{S}$.

Finally note that S is a Serre tensor-ideal if and only if it is closed under tensoring with representable functors (see Lemma 3.3.5). $\square$

Remark 3.3.7. Let $\mathcal{C}$ be a monoidal finitely accessible category with products as in Assumption 3.3.1. By Theorem 3.3.6 and Theorem 2.4.32, a definable subcategory $\mathcal{D} \subseteq \mathcal{C}$ is fp-hom-closed if and only if the collection of pp-pairs ϕ/ψ such that $\phi(X)/\psi(X) = 0$ for all $X \in \mathcal{D}$, forms a Serre tensor-ideal in the category $\mathbb{L}(\mathcal{C})^{\mathrm{eq+}}$ with the monoidal structure as discussed in Remark 3.3.2. By Lemma 3.3.5, a Serre subcategory of $\mathbb{L}(\mathcal{C})^{\mathrm{eq+}}$ is a tensor-ideal if and only if it is closed under tensoring with pp-pairs of the form $(x_C = x_C)/(x_C = 0)$ for all $C \in \mathcal{C}^{\mathrm{fp}}$. The equivalence in Theorem 2.4.32 sends a function $F_f \in \mathcal{C}^{\mathrm{fp}}\text{-mod}$ with presentation

$$(B,-) \xrightarrow{(f,-)} (A,-) \to F_f \to 0,$$

where $f : A \to B$ is a morphism in $\mathcal{C}^{\mathrm{fp}}$, to the pp-pair $(x_A = x_A)/(\exists y_B\, x_A = y_B \circ f)$. Thus for $C \in \mathcal{C}^{\mathrm{fp}}$,

$$\begin{aligned}(x_C = x_C)/(x_C = 0) \;\otimes\; &(x_A = x_A)/(\exists y_B\; x_A = y_B f)\\ &= (x_{C\otimes A} = x_{C\otimes A})/(\exists y_{C\otimes B}\; x_{C\otimes A} = y_{C\otimes B}(C\otimes f)).\end{aligned}$$

By Theorem 3.3.6, if $(\mathcal{D},\mathcal{C},\otimes)$ is an object of $\mathbb{DEF}^{\otimes}$, then the corresponding Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$-mod is a Serre tensor-ideal. Next we use this to define a monoidal structure on $\mathrm{fun}(\mathcal{D})$. We prove the following lemma first.

Lemma 3.3.8. *Let $\mathcal{C}$ be as in Assumption 3.3.1. Then*

$$(C,-)\otimes - : \mathcal{C}^{\mathrm{fp}}\text{-Mod} \to \mathcal{C}^{\mathrm{fp}}\text{-Mod}$$

is exact for all $C \in \mathcal{C}^{\mathrm{fp}}$. If we assume further that $C \in \mathcal{C}^{\mathrm{fp}}$ is rigid then so is $(C,-)$ with dual given by $(C^{\vee},-)$.

Proof. We already know that $(C, -) \otimes -$ is a left adjoint [20, Theorem 3.3 and Theorem 3.6] and therefore right exact. We show that $(C, -) \otimes -$ is also a right adjoint and therefore is an exact functor.

We first define the left adjoint $\mathbf{L}_C : \mathcal{C}^{\text{fp}}\text{-Mod} \to \mathcal{C}^{\text{fp}}\text{-Mod}$ on finitely presented functors. Given $F_f \in \mathcal{C}^{\text{fp}}\text{-mod}$ with presentation $(B, -) \xrightarrow{(f,-)} (A, -) \to F_f \to 0$, denote by $\mathbf{L}_C(F_f)$ the functor with presentation

$$(\hom(C, B), -) \xrightarrow{(\hom(C,f),-)} (\hom(C, A), -) \to \mathbf{L}_C(F_f) \to 0.$$

It can be checked that this definition does not depend on the choice of f.

Now, given another finitely presented functor F_g with presentation $(V, -) \xrightarrow{(g,-)} (U, -) \to F_g \to 0$ and a morphism $\alpha : F_f \to F_g$, chose any $\alpha_1 : U \to A$ and $\alpha_2 : V \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
0 & 0 \\
F_f \arrow[u] \arrow[r, "\alpha"] & F_g \arrow[u] \\
(A, -) \arrow[u, "\pi_f"] \arrow[r, "(\alpha_1, -)"] & (U, -) \arrow[u, "\pi_g"'] \\
(B, -) \arrow[u, "(f, -)"] \arrow[r, "(\alpha_2, -)"] & (V, -) \arrow[u, "(g, -)"']
\end{tikzcd}$$

Then define the morphism $\mathbf{L}_C(\alpha) : \mathbf{L}_C(F_f) \to \mathbf{L}_C(F_g)$ by the unique map making the following diagram commute.

$$\begin{tikzcd}
0 & 0 \\
\mathbf{L}_C(F_f) \arrow[u] \arrow[r, "\mathbf{L}_C(\alpha)"] & \mathbf{L}_C(F_g) \arrow[u] \\
(\hom(C,A),-) \arrow[u, "\pi_{\hom(C,f)}"] \arrow[r, "{(\hom(C,\alpha_1),-)}"] & (\hom(C,U),-) \arrow[u, "\pi_{\hom(C,g)}"'] \\
(\hom(C,B),-) \arrow[u, "{(\hom(C,f),-)}"] \arrow[r, "{(\hom(C,\alpha_2),-)}"] & (\hom(C,V),-) \arrow[u, "{(\hom(C,g),-)}"']
\end{tikzcd}$$

Note that this does not depend of the choice of α_1 and α_2. To define $\mathbf{L}_C$ (up to isomorphism) on any $F \in \mathcal{C}^{\mathrm{fp}}\text{-Mod}$, we assert that $\mathbf{L}_C$ commutes with direct limits.

It is easy to check that $\mathbf{L}_C : \mathcal{C}^{\mathrm{fp}}\text{-Mod} \to \mathcal{C}^{\mathrm{fp}}\text{-Mod}$ defines (up to isomorphism) a functor, indeed this follows from the fact that $\hom(C,-) : \mathcal{C} \to \mathcal{C}$ is a functor. We claim that this functor is left adjoint to $(C,-)\otimes - : \mathcal{C}^{\mathrm{fp}}\text{-Mod} \to \mathcal{C}^{\mathrm{fp}}\text{-Mod}$.

As $(C,-)\otimes -$ and $\mathbf{L}_C$ commute with direct limits, it is enough to define the unit and counit of the adjunction on finitely presented functors. Indeed, any functor $F \in \mathcal{C}^{\mathrm{fp}}\text{-Mod}$ can be expressed as a direct limit of finitely presented functors, say $F = \varinjlim_{i\in I} F_i$ where each $F_i \in \mathcal{C}^{\mathrm{fp}}\text{-mod}$. By the universal property of direct limits, the value of the unit, $\boldsymbol{\eta}_F : F \to ((C,-)\otimes \mathbf{L}_C(F)$, and the counit, $\boldsymbol{\varepsilon}_F : \mathbf{L}_C((C,-)\otimes F) \to F$, at F, is uniquely determined by the respective components at the F_i.

The unit, $\boldsymbol{\eta} : \mathrm{Id}_{\mathcal{C}^{\mathrm{fp}}\text{-Mod}} \to ((C,-)\otimes -)\circ \mathbf{L}_C$, is defined on finitely presented functors as follows. For $F_f \in \mathcal{C}^{\mathrm{fp}}\text{-mod}$ we define $\boldsymbol{\eta}_{F_f} : F_f \to (C,-)\otimes \mathbf{L}_C(F_f)$ as the unique map such that the following diagram commutes, where $\varepsilon_C : (C\otimes -)\circ \hom(C,-) \to \mathrm{Id}_{\mathcal{C}}$ is the counit of the adjunction between $C\otimes -$ and $\hom(C,-)$.

$$\begin{tikzcd}
0 & 0 \\
F_f \arrow[u] \arrow[r, "\boldsymbol{\eta}_{F_f}"] & (C,-)\otimes \mathbf{L}_C(F_f) \arrow[u] \\
(A,-) \arrow[u, "\pi_f"] \arrow[r, "((\varepsilon_C)_A,-)"] & (C\otimes \hom(C,A),-) \arrow[u, "(C,-)\otimes \pi_{\hom(C,f)}"'] \\
(B,-) \arrow[u, "(f,-)"] \arrow[r, "((\varepsilon_C)_B,-)"] & (C\otimes \hom(C,B),-) \arrow[u, "(C\otimes \hom(C,f),-)"']
\end{tikzcd}$$

Similarly we define the counit of the adjunction, $\boldsymbol{\varepsilon} : \mathbf{L}_C \circ (C,-)\otimes - \to \mathrm{Id}_{\mathcal{C}^{\mathrm{fp}}\text{-Mod}}$, as follows. For $F_f \in \mathcal{C}^{\mathrm{fp}}$-mod we define $\boldsymbol{\varepsilon}_{F_f} : \mathbf{L}_C((C,-))\otimes F_f) \to F_f$ as the unique map such that the following diagram commutes, where $\eta_C : \mathrm{Id}_{\mathcal{C}} \to \hom(C,-)\circ (C\otimes -)$ is the unit of the adjunction between $C\otimes -$ and $\hom(C,-)$.

$$\begin{tikzcd}
0 & 0 \\
\mathbf{L}_C((C,-)\otimes F_f) \arrow[u] \arrow[r, "\boldsymbol{\varepsilon}_{F_f}"] & F_f \arrow[u] \\
(C\otimes \hom(C,A),-) \arrow[u, "\pi_{\hom(C,C\otimes f)}"] \arrow[r, "((\eta_C)_A,-)"] & (A,-) \arrow[u, "\pi_f"'] \\
(C\otimes \hom(C,B),-) \arrow[u, "(\hom(C,C\otimes f),-)"] \arrow[r, "((\eta_C)_B,-)"] & (B,-) \arrow[u, "(f,-)"']
\end{tikzcd}$$

It can be seen that $\boldsymbol{\eta}_{F_f}$ and $\boldsymbol{\varepsilon}_{F_f}$ don't depend of the choice of presentation. It remains to check that the triangle identities hold. Again, it is enough to check when evaluating at finitely presented functors and these follow easily from the triangle identities on the adjunction between $C \otimes -$ and $\hom(C, -)$.

If, in addition, we assume that $C \in \mathcal{C}^{\mathrm{fp}}$ is rigid then $\hom(C, -) \cong C^{\vee} \otimes -$ and therefore $\mathbf{L}_C \cong (C^{\vee}, -) \otimes -$. Thus, $(C^{\vee}, -) \otimes -$ is left adjoint to $(C, -) \otimes -$ and $(C, -) \otimes - \cong ((C^{\vee})^{\vee}, -) \otimes -$ is left adjoint to $(C^{\vee}, -) \otimes -$. Therefore, $(C, -) \otimes -$ is rigid with dual given by $(C^{\vee}, -) \otimes -$ as required. $\square$

Next we define an additive symmetric monoidal structure on $\mathrm{fun}(\mathcal{D})$.

Definition 3.3.9. Suppose $(\mathcal{D}, \mathcal{C}, \otimes) \in \mathbb{DEF}^{\otimes}$ and let $\mathsf{S} \subseteq \mathcal{C}^{\mathrm{fp}}\text{-mod}$ be the Serre subcategory corresponding to $\mathcal{D}$. By Theorem 3.3.6, S is a Serre tensor-ideal of $\mathcal{C}^{\mathrm{fp}}$-mod. First we define an additive symmetric monoidal structure on $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$.

By [21], if the multiplicative system Σ_{S} of all the morphisms α in $\mathcal{C}^{\mathrm{fp}}$-mod such that $\ker(\alpha)$, $\mathrm{coker}(\alpha) \in \mathsf{S}$ is closed under tensoring with objects of $\mathcal{C}^{\mathrm{fp}}$-mod, then $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ has a monoidal structure such that the localisation functor $q : \mathcal{C}^{\mathrm{fp}}\text{-mod} \to \mathrm{fun}(\mathcal{D}) = \mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ is universal among *monoidal* functors which map the morphisms in Σ_{S} to isomorphisms.

By Lemma 3.3.8, for any $C \in \mathcal{C}^{\mathrm{fp}}$, $(C, -) \otimes - : \mathcal{C}^{\mathrm{fp}}\text{-Mod} \to \mathcal{C}^{\mathrm{fp}}\text{-Mod}$ is exact. Since, $\mathcal{C}^{\mathrm{fp}}$-mod is an abelian subcategory of $\mathcal{C}^{\mathrm{fp}}$-Mod, $(C, -) \otimes - : \mathcal{C}^{\mathrm{fp}}\text{-mod} \to \mathcal{C}^{\mathrm{fp}}\text{-mod}$ is also exact. Consequently, for every $\alpha : F \to G$ in $\mathcal{C}^{\mathrm{fp}}$-mod, $\ker((C, -) \otimes \alpha) \cong (C, -) \otimes \ker(\alpha)$ and $\mathrm{coker}((C, -) \otimes \alpha) \cong (C, -) \otimes \mathrm{coker}(\alpha)$. Therefore if S is a tensor-ideal and $\alpha \in \Sigma_{\mathsf{S}}$ then $\ker((C, -) \otimes \alpha)$, $\mathrm{coker}((C, -) \otimes \alpha) \in \mathsf{S}$ so $(C, -) \otimes \alpha \in \Sigma_{\mathsf{S}}$.

Now consider the morphism $F_g \otimes \alpha$. We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
(V,-) \otimes F \arrow[r, "(g,-) \otimes F"] \arrow[d, "(V,-) \otimes \alpha"] & (U,-) \otimes F \arrow[r] \arrow[d, "(U,-) \otimes \alpha"] & F_g \otimes F \arrow[r] \arrow[d, dashed, "F_g \otimes \alpha"] & 0 \\
(V,-) \otimes G \arrow[r, "(g,-) \otimes G"] & (U,-) \otimes G \arrow[r] & F_g \otimes G \arrow[r] & 0
\end{tikzcd}$$

Since for every $D \in \mathcal{D}$, $((V, -) \otimes \alpha)_D$ and $((U, -) \otimes \alpha)_D$ are isomorphisms, so is $(F_g \otimes \alpha)_D$. Hence $\ker(F_g \otimes \alpha)$, $\mathrm{coker}(F_g \otimes \alpha) \in \mathsf{S}$ and $F_g \otimes \alpha \in \Sigma_{\mathsf{S}}$.

Applying [21, Corollary 1.4] we get an additive symmetric monoidal structure on $\mathcal{C}^{\mathrm{fp}}$-mod/$\mathsf{S}$. We induce a monoidal structure on $\mathrm{fun}(\mathcal{D}) = (\mathcal{D}, \mathbf{Ab})^{\Pi\rightarrow}$ via the equivalence given in Theorem 2.4.37.

Next we show that, if $\mathcal{D}$ satisfies the exactness criterion, the monoidal structure on $\mathrm{fun}(\mathcal{D})$ is exact in each variable.

Proposition 3.3.10. *Let $\mathcal{C}$ be as in Assumption 3.3.1. Suppose $\mathcal{D}$ is an fp-hom-closed definable subcategory and induce a monoidal structure on* $\mathrm{fun}(\mathcal{D})$ *as in Definition 3.3.9.*

If $\mathcal{D}$ satisfies the exactness criterion then the monoidal structure on $\mathrm{fun}(\mathcal{D})$ *is exact in each variable.*

Proof. Suppose $\mathcal{D}$ satisfies the exactness criterion, i.e. for any $f : A \to B$ and $g : U \to V$ in $\mathcal{C}^{\mathrm{fp}}$ and for any $D \in \mathcal{D}$, if $h : A \otimes U \to D$ satisfies $(f \otimes U)|h$ and $(A \otimes g)|h$ then $(f \otimes g)|h$. Suppose further that $0 \to F \to G \to L \to 0$ is an exact sequence in $\mathrm{fun}(\mathcal{D})$. It is (isomorphic to) the image of an exact sequence $0 \to F_f \xrightarrow{\alpha} F_g \xrightarrow{\beta} F_l \to 0$ in $\mathcal{C}^{\mathrm{fp}}$-mod (see [48, Lemma 11.1.6 and Corollary 11.1.42]). If $K \in \mathrm{fun}(\mathcal{D})$ then K is isomorphic to the image of F_k for some $F_k \in \mathcal{C}^{\mathrm{fp}}$-mod. Therefore, since the localisation functor is monoidal, showing that $0 \to K \otimes F \xrightarrow{K\otimes\alpha} K \otimes G \xrightarrow{K\otimes\beta} K \otimes L \to 0$ is a short exact sequence in $\mathrm{fun}(\mathcal{D})$ is equivalent to showing that the image of the (not necessarily exact) sequence

$$0 \to F_k \otimes F_f \to F_k \otimes F_g \to F_k \otimes F_l \to 0$$

under the localisation functor gives a short exact sequence. By [49, Theorem 12.10], this is equivalent to showing that

$$0 \to (\overrightarrow{F_k \otimes F_f})(D) \to (\overrightarrow{F_k \otimes F_g})(D) \to (\overrightarrow{F_k \otimes F_l})(D) \to 0$$

is exact for all $D \in \mathcal{D}$.

Suppose F_k has presentation $(T, -) \xrightarrow{(k,-)} (S, -) \to F_k \to 0$, where $k : S \to T$ is a morphism in $\mathcal{C}^{\mathrm{fp}}$, then we have the following commutative diagram in $\mathcal{C}^{\mathrm{fp}}$-mod.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & F_k \otimes F_f \arrow[u] \arrow[r, "F_k \otimes \alpha"] & F_k \otimes F_g \arrow[u] \arrow[r, "F_k \otimes \beta"] & F_k \otimes F_l \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & (S,-) \otimes F_f \arrow[u, "\pi_k \otimes F_f"] \arrow[r, "(S,-)\otimes \alpha"] & (S,-) \otimes F_g \arrow[u] \arrow[r, "(S,-)\otimes \beta"] & (S,-) \otimes F_l \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & (T,-) \otimes F_f \arrow[u, "(k,-)\otimes F_f"] \arrow[r, "(T,-)\otimes \alpha"'] & (T,-) \otimes F_g \arrow[u, "(k,-)\otimes F_g"] \arrow[r, "(T,-)\otimes \beta"'] & (T,-) \otimes F_l \arrow[u, "(k,-)\otimes F_l"'] \arrow[r] & 0
\end{tikzcd}$$

Here the second row is exact since $F_k \otimes -$ is a left adjoint and therefore right exact. The third and fourth rows are exact by Lemma 3.3.8. We must show that $\overrightarrow{(F_k \otimes \alpha)}_D$ is a monomorphism (or has zero kernel) for all $D \in \mathcal{D}$. Fix $D \in \mathcal{D}$. To enhance readability, for a functor $F \in \mathcal{C}^{\mathrm{fp}}$-mod we will suppress the usual notation, $\overrightarrow{F}$, for the unique extension to a functor $\mathcal{C} \to \mathbf{Ab}$ which commutes with direct limits, and simply use F. By Yoneda's lemma, $(F_k \otimes \alpha)_D$ is a monomorphism if and only if, for every morphism $\beta : (D,-) \to F_k \otimes F_f$, if $(F_k \otimes \alpha) \circ \beta = 0$ then $\beta = 0$. Suppose $\beta : (D,-) \to F_k \otimes F_f$ satisfies $(F_k \otimes \alpha) \circ \beta = 0$.

Let us fix some notation. Say that F_f has presentation $(B,-) \xrightarrow{(f,-)} (A,-) \xrightarrow{\pi_f} F_f \to 0$ and F_g has presentation $(V,-) \xrightarrow{(g,-)} (U,-) \xrightarrow{\pi_g} F_g \to 0$. Choose any morphisms $\alpha_1 : U \to A$ and $\alpha_2 : V \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
(B,-) \arrow[r, "(f,-)"] \arrow[d, "(\alpha_2,-)"'] & (A,-) \arrow[r, "\pi_f"] \arrow[d, "(\alpha_1,-)"'] & F_f \arrow[r] \arrow[d, "\alpha"'] & 0 \\
(V,-) \arrow[r, "(g,-)"] & (U,-) \arrow[r, "\pi_g"] & F_g \arrow[r] & 0
\end{tikzcd}$$

The proof will proceed in the following steps.

Step 1: We will show that $\beta = (\pi_k \otimes \pi_f) \circ (\gamma_1, -)$ where $\gamma_1 : S \otimes A \to D$ and there exists $l' : S \otimes V \to D$ and $\xi_1 : T \otimes U \to D$ such that

$$\gamma_1 \circ (S \otimes \alpha_1) - l' \circ (S \otimes g) = \xi_1 \circ (k \otimes U).$$

Step 2: We will use the exactness criterion to conclude that

$$\gamma_1 \circ (S \otimes \alpha_1) - l' \circ (S \otimes g) = y_1 \circ (k \otimes g) + y_2 \circ (k \otimes \alpha_1),$$

where $y_1 : T \otimes V \to D$ and $y_2 : T \otimes A \to D$.

Step 3: We show that

$$((S,-) \otimes \alpha) \circ ((S,-) \otimes \pi_f) \circ (\gamma_1, -) = ((S,-) \otimes \alpha) \circ ((S,-) \otimes \pi_f) \circ (y_2 \circ (k \otimes A), -)$$

and use that $((S,-) \otimes \alpha)$ is a monomorphism to conclude that

$$((S,-) \otimes \pi_f) \circ (\gamma_1, -) = ((S,-) \otimes \pi_f) \circ (y_2 \circ (k \otimes A), -).$$

Step 4: We compose both sides of

$$((S,-) \otimes \pi_f) \circ (\gamma_1, -) = ((S,-) \otimes \pi_f) \circ (k \otimes A, -) \circ (y_2, -)$$

by $\pi_k \otimes F_f$ to get $\beta = 0$.

Step 1: Since $\pi_k \otimes F_f$ is an epimorphism and $(D,-)$ is projective, there exists some $\gamma : (D,-) \to (S,-) \otimes F_f$ such that $\beta = (\pi_k \otimes F_f) \circ \gamma$.

Since $0 = (F_k \otimes \alpha) \circ \beta = (F_k \otimes \alpha) \circ (\pi_k \otimes F_f) \circ \gamma = (\pi_k \otimes F_g) \circ ((S,-) \otimes \alpha) \circ \gamma$ and $(D,-)$ is projective, $\zeta := ((S,-) \otimes \alpha) \circ \gamma : (D,-) \to (S,-) \otimes F_g$ factors via $(k,-) \otimes F_g$, say $\zeta = ((k,-) \otimes F_g) \circ \xi$, where $\xi : (D,-) \to (T,-) \otimes F_g$ (see the diagram below).

$$\begin{tikzcd}
(T \otimes V, -) \arrow[d, "{(T \otimes g, -)}"'] & & & \\
(T \otimes U, -) \arrow[d] & (D, -) \arrow[l, dashed, "{(\xi_1, -)}"'] \arrow[dl, dashed, "\xi"'] \arrow[d, "\zeta"'] \arrow[dr, "0"] & & \\
(T, -) \otimes F_g \arrow[r, "{(k, -) \otimes F_g}"] \arrow[d] & (S, -) \otimes F_g \arrow[r, "\pi_k \otimes F_g"] & F_k \otimes F_g \arrow[r] & 0 \\
0 & & &
\end{tikzcd}$$

As $(D, -)$ is projective, ξ factors via $(T, -) \otimes \pi_g$, say $\xi = ((T, -) \otimes \pi_g) \circ (\xi_1, -)$ where $\xi_1 : T \otimes U \to D$. Similarly, $\gamma : (D, -) \to (S, -) \otimes F_f$ factors via $(S, -) \otimes \pi_f$, say $\gamma = ((S, -) \otimes \pi_f) \circ (\gamma_1, -)$ where $\gamma_1 : S \otimes A \to D$. Therefore, as $((S, -) \otimes \alpha) \circ \gamma = \zeta = ((k, -) \otimes F_g) \circ \xi$ we have

$$((S, -) \otimes \pi_g) \circ (S \otimes \alpha_1, -) \circ (\gamma_1, -) = ((S, -) \otimes \pi_g) \circ (k \otimes U, -) \circ (\xi_1, -).$$

Set $l = \gamma_1 \circ (S \otimes \alpha_1) - \xi_1 \circ (k \otimes U) : S \otimes U \to D$. Then

$$((S, -) \otimes \pi_g) \circ (l, -) = 0,$$

meaning $(l, -)$ factors via $(S \otimes g, -)$, i.e. $l = l' \circ (S \otimes g)$ for some $l' : S \otimes V \to D$.

We have shown that

$$l = \gamma_1 \circ (S \otimes \alpha_1) - \xi_1 \circ (k \otimes U) = l' \circ (S \otimes g).$$

Rearranging we have

$$\gamma_1 \circ (S \otimes \alpha_1) - l' \circ (S \otimes g) = \xi_1 \circ (k \otimes U).$$

Step 2: Now we can use the exactness criterion. Set $h := \gamma_1 \circ (S \otimes \alpha_1) - l' \circ (S \otimes g) = \xi_1 \circ (k \otimes U)$. Consider the morphism $(g, \alpha_1) : U \to V \oplus A$ such that $p_1 \circ (g, \alpha_1) = g$ and $p_2 \circ (g, \alpha_1) = \alpha_1$, where p_1 and p_2 denote the projection maps. Then $(S \otimes (g, \alpha_1))|h$ and $(k \otimes U)|h$ so since the exactness criterion holds for $\mathcal{D}$, $(k \otimes (g, \alpha_1))|h$ and there exists some $y : T \otimes (V \oplus A) \to D$ such that $y \circ (k \otimes (g, \alpha_1)) = h$. Set $y_1 = y \circ (T \otimes i_1)$ and $y_2 = y \circ (T \otimes i_2)$, where $i_1 : V \to V \oplus A$ and $i_2 : A \to V \oplus A$ are the inclusion maps. Then we have

$$\gamma_1 \circ (S \otimes \alpha_1) - l' \circ (S \otimes g) = h = y_1 \circ (k \otimes g) + y_2 \circ (k \otimes \alpha_1).$$

Step 3: Composing $(h, -)$ with $(S, -) \otimes \pi_g$ we get,

$$\begin{aligned} ((S, -) \otimes \pi_g) \circ (\gamma_1 \circ (S \otimes \alpha_1), -) &= ((S, -) \otimes \pi_g) \circ (h, -) \\ &= ((S, -) \otimes \pi_g) \circ (y_2 \circ (k \otimes \alpha_1), -). \end{aligned}$$

$$\begin{tikzcd} (S \otimes B, -) \arrow[r, "(S \otimes f, -)"] \arrow[d, "(S \otimes \alpha_2, -)"'] & (S \otimes A, -) \arrow[r, "(S, -) \otimes \pi_f"] \arrow[d, "(S \otimes \alpha_1, -)"'] & (S, -) \otimes F_f \arrow[r] \arrow[d, "(S, -) \otimes \alpha"'] & 0 \\ (S \otimes V, -) \arrow[r, "(S \otimes g, -)"] & (S \otimes U, -) \arrow[r, "(S, -) \otimes \pi_g"] & (S, -) \otimes F_g \arrow[r] & 0 \end{tikzcd}$$

As shown on the commutative diagram above, we have

$$((S, -) \otimes \pi_g) \circ ((S, -) \otimes (\alpha_1, -)) = ((S, -) \otimes \alpha) \circ ((S, -) \otimes \pi_f).$$

Therefore, as

$$((S, -) \otimes \pi_g) \circ (\gamma_1 \circ (S \otimes \alpha_1), -) = ((S, -) \otimes \pi_g) \circ (y_2 \circ (k \otimes \alpha_1), -)$$

we have

$$((S, -) \otimes \alpha) \circ ((S, -) \otimes \pi_f) \circ (\gamma_1, -) = ((S, -) \otimes \alpha) \circ ((S, -) \otimes \pi_f) \circ (y_2 \circ (k \otimes A), -).$$

Since $(S,-)\otimes\alpha$ is a monomorphism we have

$$((S,-)\otimes\pi_f)\circ(\gamma_1,-) = ((S,-)\otimes\pi_f)\circ(y_2\circ(k\otimes A),-).$$

Step 4: Finally note that

$$((S,-)\otimes\pi_f)\circ(y_2\circ(k\otimes A),-) = ((k,-)\otimes F_f)\circ((T,-)\otimes\pi_f)\circ(y_2,-)$$

and therefore, as

$$\gamma = ((S,-)\otimes\pi_f)\circ(\gamma_1,-) = ((k,-)\otimes F_f)\circ((T,-)\otimes\pi_f)\circ(y_2,-)$$

we get that

$$\beta = (\pi_k\otimes F_f)\circ\gamma = (\pi_k\otimes F_f)\circ((k,-)\otimes F_f)\circ((T,-)\otimes\pi_f)\circ(y_2,-) = 0,$$

as required. $\square$

In fact, the converse of Proposition 3.3.10 is also true.

Proposition 3.3.11. *Let $\mathcal{C}$ be as in Assumption 3.3.1. Suppose $\mathcal{D}$ is an fp-hom-closed definable subcategory and induce a monoidal structure on* $\mathrm{fun}(\mathcal{D})$ *as in Definition 3.3.9. If the monoidal structure on* $\mathrm{fun}(\mathcal{D})$ *is exact in each variable then $\mathcal{D}$ satisfies the exactness criterion.*

Proof. Suppose that the induced monoidal structure on $\mathrm{fun}(\mathcal{D})$ is exact in each variable. Suppose $f : A \to B$ and $g : U \to V$ are morphisms in $\mathcal{C}^{\mathrm{fp}}$. By [49, Corollary 3.11], $\mathcal{C}^{\mathrm{fp}}$ has weak cokernels. Let $g' : V \to W$ be a weak cokernel of g.

Then $(W,-) \xrightarrow{(g',-)} (V,-) \xrightarrow{(g,-)} (U,-)$ is exact in $\mathcal{C}^{\mathrm{fp}}$-mod, that is $\mathrm{im}((g',-)) = \ker((g,-))$. Therefore, its image $(W,-)_{\mathsf{S}} \xrightarrow{(g',-)_{\mathsf{S}}} (V,-)_{\mathsf{S}} \xrightarrow{(g,-)_{\mathsf{S}}} (U,-)_{\mathsf{S}}$ is exact in $\mathrm{fun}(\mathcal{D})$ and by assumption, $(F_f)_{\mathsf{S}}\otimes(W,-)_{\mathsf{S}} \xrightarrow{(F_f)_{\mathsf{S}}\otimes(g',-)_{\mathsf{S}}} (F_f)_{\mathsf{S}}\otimes(V,-)_{\mathsf{S}} \xrightarrow{(F_f)_{\mathsf{S}}\otimes(g,-)_{\mathsf{S}}} (F_f)_{\mathsf{S}}\otimes(U,-)_{\mathsf{S}}$ is also exact in $\mathrm{fun}(\mathcal{D})$. As the localisation functor is monoidal, this is equivalent to, $(F_f\otimes(W,-))(D) \xrightarrow{(F_f\otimes g',-))_D} (F_f\otimes(V,-))(D) \xrightarrow{(F_f\otimes(g,-))_D} (F_f\otimes(U,-))(D)$ being exact in **Ab** for all $D\in\mathcal{D}$, by [49, Theorem 12.10]. Consider the diagram below.

$$\begin{tikzcd}
0 & 0 & 0 \\
F_f \otimes (W,-) \arrow[u] \arrow[r, "F_f \otimes (g',-)"] & F_f \otimes (V,-) \arrow[u] \arrow[r, "F_f \otimes (g,-)"] & F_f \otimes (U,-) \arrow[u] \\
(A \otimes W,-) \arrow[u, "\pi_f \otimes (W,-)"] \arrow[r, "(A \otimes g',-)"] & (A \otimes V,-) \arrow[u, "\pi_f \otimes (V,-)"] \arrow[r, "(A \otimes g,-)"] & (A \otimes U,-) \arrow[u, "\pi_f \otimes (U,-)"'] \\
(B \otimes W,-) \arrow[u, "(f \otimes W,-)"] \arrow[r, "(B \otimes g',-)"] & (B \otimes V,-) \arrow[u, "(f \otimes V,-)"] \arrow[r, "(B \otimes g,-)"] & (B \otimes U,-) \arrow[u, "(f \otimes U,-)"']
\end{tikzcd}$$

Given any $h : A \otimes U \to D$ such that $(f \otimes U)|h$ and $(A \otimes g)|h$ there exists $h_1 : B \otimes U \to D$ such that $h = h_1 \circ (f \otimes U)$ and $h_2 : A \otimes V \to D$ such that $h = h_2 \circ (A \otimes g)$. But this means that

$$\begin{aligned}
((F_f \otimes (g,-)) \circ (\pi_f \otimes (V,-)))_D(h_2) &= (\pi_f \otimes (U,-))_D(h_2 \circ (A \otimes g)) \\
&= (\pi_f \otimes (U,-))_D(h_1 \circ (f \otimes U)) \\
&= 0.
\end{aligned}$$

So $(\pi_f \otimes (V,-))_D(h_2)$ is in the kernel of $(F_f \otimes (g,-))_D$ which is equal to the image of $(F_f \otimes (g',-))_D$. Therefore there exists some $z \in (F_f \otimes (W,-))(D)$ such that $(F_f \otimes (g',-))_D(z) = (\pi_f \otimes (V,-))_D(h_2)$. But then, since $(\pi_f \otimes (W,-))_D$ is an epimorphism in **Ab**, there exists some morphism $z' : A \otimes W \to D$ such that $(\pi_f \otimes (W,-))_D(z') = z$.

Next, notice that $(\pi_f \otimes (V,-))_D(h_2 - (z' \circ (A \otimes g'))) = 0$. So there exists some $y : B \otimes V \to D$ such that $y \circ (f \otimes V) = h_2 - (z' \circ (A \otimes g'))$. But then $y \circ (f \otimes g) = y \circ (f \otimes V) \circ (A \otimes g) = h_2 \circ (A \otimes g) = h$. That is $(f \otimes g)|h$, as required. □

Theorem 3.3.12. *There exists a 2-functor $\theta : (\mathbb{DEF}^{\otimes})^{\mathrm{op}} \to \mathbb{ABEX}^{\otimes}$ which maps $(\mathcal{D}, \mathcal{C}, \otimes) \in \mathbb{DEF}^{\otimes}$ to $\mathrm{fun}(\mathcal{D}) = (\mathcal{D}, \mathbf{Ab})^{\Pi\to} \in \mathbb{ABEX}^{\otimes}$. The exact additive symmetric monoidal structure on $\mathrm{fun}(\mathcal{D}) = (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ is given by inducing Day convolution product on $\mathcal{C}^{\mathrm{fp}}\text{-mod}$, establishing a monoidal structure on the localisation $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ such that the quotient functor $q : \mathcal{C}^{\mathrm{fp}}\text{-mod} \to \mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S}$ is monoidal and asserting that the equivalence $\mathcal{C}^{\mathrm{fp}}\text{-mod}/\mathsf{S} \simeq (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ is monoidal (Definition 3.3.9).*

On 1-morphisms θ maps $I : \mathcal{D} \to \mathcal{D}'$ in $\mathbb{DEF}^{\otimes}$ to the 1-morphism $I_0 : (\mathcal{D}', \mathbf{Ab})^{\Pi\to} \to (\mathcal{D}, \mathbf{Ab})^{\Pi\to}$ in $\mathbb{ABEX}^{\otimes}$ where I_0 maps a functor $F : \mathcal{D}' \to \mathbf{Ab}$ to $F \circ I : \mathcal{D} \to \mathbf{Ab}$.

Given a 2-morphisms $\tau : I \to J$ in $\mathbb{DEF}^{\otimes}$, $\theta(\tau) : I_0 \to J_0$ is the natural transformation where the component at $F \in (\mathcal{D}', \mathbf{Ab})^{\Pi\to}$ is the natural transformation

$$\theta(\tau)_F : I_0(F) \to J_0(F)$$

with component at $X \in \mathcal{D}$ given by

$$F(I(X)) \xrightarrow{F(\tau_X)} F(J(X)),$$

noting that $I_0(F)(X) = (F \circ I)(X) = F(I(X))$.

Proof. θ is well defined on objects by Theorem 3.3.6 and Proposition 3.3.10. Given a morphism $I : \mathcal{D} \to \mathcal{D}'$ in $\mathbb{DEF}^{\otimes}$, I_0 is monoidal by definition of the morphisms in $\mathbb{DEF}^{\otimes}$ and therefore I_0 is a 1-morphism in $\mathbb{ABEX}^{\otimes}$. On natural transformations θ acts as in the original anti-equivalence and therefore θ satisfies the necessary axioms to be a 2-functor. □

3.4 The 2-functor $\xi : (\mathbb{ABEX}^{\otimes})^{\mathrm{op}} \to \mathbb{DEF}^{\otimes}$

Next we define a 2-functor $\xi : (\mathbb{ABEX}^{\otimes})^{\mathrm{op}} \to \mathbb{DEF}^{\otimes}$ which maps a skeletally small abelian category $\mathscr{A}$ with an exact additive symmetric monoidal structure to $(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}), \mathscr{A}\text{-Mod}, \otimes)$, where $\otimes$ is induced by Day convolution product.

First we show that $(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}), \mathscr{A}\text{-Mod}, \otimes)$ is a well-defined object of $\mathbb{DEF}^{\otimes}$ (see Theorem 3.4.2).

Lemma 3.4.1. *Let $\mathscr{A}$ be an additive symmetric monoidal, skeletally small abelian category. Suppose that for every exact functor $E : \mathscr{A} \to \mathbf{Ab}$, every $X \in \mathscr{A}$ and every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$,*

$$0 \to E(X \otimes A) \to E(X \otimes B) \to E(X \otimes C) \to 0$$

is exact in $\mathbf{Ab}$. Then the monoidal structure on $\mathscr{A}$ is exact in each variable.

Proof. By Freyd-Mitchell Embedding Theorem [25, Theorem 7.34] there exists a ring R and an exact fully faithful functor $F : \mathscr{A} \to R\text{-Mod}$. Composing F with the forgetful functor $R\text{-Mod} \to \mathbf{Ab}$ we get an exact faithful functor $E : \mathscr{A} \to \mathbf{Ab}$. For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$,

$$0 \to E(X \otimes A) \to E(X \otimes B) \to E(X \otimes C) \to 0$$

is exact in $\mathbf{Ab}$ but as E is faithful, E reflects exactness, so

$$0 \to X \otimes A \to X \otimes B \to X \otimes C \to 0$$

is exact in $\mathscr{A}$ as required. $\square$

Theorem 3.4.2. *Let $\mathscr{A}$ be an additive symmetric monoidal, skeletally small abelian category. The following are equivalent:*

(i) The definable subcategory $\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}) \subseteq \mathscr{A}\text{-Mod}$ is fp-hom-closed (with respect to Day convolution product).

(ii) The Serre subcategory $\mathsf{S}_{\mathrm{Ex}} \subseteq (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ of all functors F such that $\overrightarrow{F}(E) = 0$ for all $E \in \mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ is a tensor-ideal of $(\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ (with respect to Day convolution product).

(iii) The tensor product on $\mathscr{A}$ is exact in each variable.

Proof. **(i)** $\leftrightarrow$ **(ii)**: Follows directly from Theorem 3.3.6.

(iii) $\to$ **(i)**: Suppose the monoidal structure on $\mathscr{A}$ is exact in each variable. We first show that $\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ is closed under $\hom(M, -)$ where $M \in \mathscr{A}$-mod is representable, say $M = (X, -)$. Indeed, in this case, for all $A \in \mathscr{A}$ and $E \in \mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$,

$$\hom((X,-),E)(A) \cong ((A,-), \hom((X,-),E)) \cong ((A \otimes X, -), E) \cong E(A \otimes X),$$

by the Yoneda lemma and adjunction isomorphisms. What's more, all these isomorphisms are natural in A. Therefore,

$$0 \to \hom((X,-),E)(A) \to \hom((X,-),E)(B) \to \hom((X,-),E)(C) \to 0$$

is exact if and only if

$$0 \to E(A \otimes X) \to E(B \otimes X) \to E(C \otimes X) \to 0$$

is exact. But the latter statement holds by our assumption on $\mathscr{A}$, as E is an exact functor. Therefore $\hom((X,-),E)$ is exact as required.

Now we generalise to $F_f \in \mathscr{A}$-mod. We want to show that $\hom(F_f, E) : \mathscr{A} \to \mathbf{Ab}$ is an exact functor.

First note that $(F_f, -)|_{\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})}$ commutes with direct products and direct limits and therefore is an object of $\mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$. By [51, Theorem 2.2], there exists an equivalence $\mathscr{A} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ given by $A \mapsto \mathrm{ev}_A$, where $\mathrm{ev}_A : \mathrm{Ex}(\mathscr{A}, \mathbf{Ab}) \to \mathbf{Ab}$ maps an exact functor E to $E(A)$. Therefore, there exists some $X_F \in \mathscr{A}$ such that $(F_f, -)|_{\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})} \cong \mathrm{ev}_{X_F}$.

Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathscr{A}$. As E is an exact functor and the monoidal structure on $\mathscr{A}$ is exact in each variable,

$$0 \to E(A \otimes X_F) \to E(B \otimes X_F) \to E(C \otimes X_F) \to 0,$$

is exact in **Ab**. As a result, by the Yoneda lemma,

$$0 \to ((A \otimes X_F, -), E) \to ((B \otimes X_F, -), E) \to ((C \otimes X_F, -), E) \to 0,$$

is exact in **Ab** and by the adjunction isomorphism this gives the exact sequence

$$\begin{aligned} 0 \to ((X_F, -), \mathrm{hom}((A, -), E)) &\to ((X_F, -), \mathrm{hom}((B, -), E)) \\ &\to ((X_F, -), \mathrm{hom}((C, -), E)) \to 0. \end{aligned}$$

Applying the Yoneda lemma once more we have the exact sequence

$$\begin{aligned} 0 \to (\mathrm{hom}((A, -), E))(X_F) &\to (\mathrm{hom}((B, -), E))(X_F) \\ &\to (\mathrm{hom}((C, -), E))(X_F) \to 0, \end{aligned}$$

which is isomorphic to

$$0 \to (F_f, \mathrm{hom}((A, -), E)) \to (F_f, \mathrm{hom}((B, -), E)) \to (F_f, \mathrm{hom}((C, -), E)) \to 0,$$

as we have already seen that $\mathrm{hom}((A, -), E)$, $\mathrm{hom}((B, -), E)$ and $\mathrm{hom}((C, -), E)$ are exact functors and $(F_f, -)|_{\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})} \cong \mathrm{ev}_{X_F}$.

Again, by the Yoneda lemma and adjunction isomorphisms we have for every $A \in \mathscr{A}$,

$$(F_f, \mathrm{hom}((A, -), E)) \cong (F_f \otimes (A, -), E) \cong ((A, -), \mathrm{hom}(F_f, E)) \cong \mathrm{hom}(F_f, E)(A).$$

What's more, all these isomorphisms are natural in A. Therefore

$$0 \to \mathrm{hom}(F_f, E)(A) \to \mathrm{hom}(F_f, E)(B) \to \mathrm{hom}(F_f, E)(C) \to 0,$$

is exact in **Ab** and $\mathrm{hom}(F_f, E)$ is an exact functor as required.

(i) $\to$ (iii) Suppose $E \in \mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ and $X \in \mathscr{A}$. By (i) we have that $\mathrm{hom}((X, -), E)$ is exact. Therefore, for all short exact sequences $0 \to A \to B \to$

$C \to 0$ in $\mathscr{A}$,

$$0 \to \hom((X,-),E)(A) \to \hom((X,-),E)(B) \to \hom((X,-),E)(C) \to 0,$$

is exact. But we have isomorphisms $\hom((X,-),E)(A) \cong ((A,-),\hom((X,-),E)) \cong ((X,-)\otimes(A,-),E) = ((X\otimes A,-),E) \cong E(X\otimes A)$ which are natural in A. Therefore, $0 \to E(X\otimes A) \to E(X\otimes B) \to E(X\otimes C) \to 0$ is also exact.

So for any exact sequence, $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, $0 \to X\otimes A \to X\otimes B \to X\otimes C \to 0$ has exact image in $\mathbf{Ab}$ under any exact functor $E : \mathscr{A} \to \mathbf{Ab}$. Lemma 3.4.1 completes the proof. $\square$

Remark 3.4.3. Recall that the objects of the 2-category $\mathbb{ABEX}^{\otimes}$ are skeletally small abelian categories with additive symmetric monoidal structures which are *exact* in each variable. However, in most examples (for instance $\mathscr{A} = R$-mod for R a commutative ring) the monoidal structure is only right exact. Theorem 3.4.2 shows where the equivalence fails without the exactness assumption. Indeed, if we desire the equivalence $\mathscr{A} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A},\mathbf{Ab}))$ to be monoidal, the Serre subcategory S_{Ex} must be a tensor-ideal, in order to induce a monoidal structure on $\mathrm{fun}(\mathrm{Ex}(\mathscr{A},\mathbf{Ab}))$.

Theorem 3.4.4. *There exists a 2-functor $\xi : (\mathbb{ABEX}^{\otimes})^{\mathrm{op}} \to \mathbb{DEF}^{\otimes}$ given on objects by $\mathscr{A} \mapsto (\mathrm{Ex}(\mathscr{A},\mathbf{Ab}), \mathscr{A}\text{-Mod}, \otimes)$ where the monoidal structure on $\mathscr{A}$-Mod is induced by Day convolution product.*

ξ maps a 1-morphism $E : \mathscr{A} \to \mathscr{A}'$ in $\mathbb{ABEX}^{\otimes}$ to the functor $E^* : \mathrm{Ex}(\mathscr{A}',\mathbf{Ab}) \to \mathrm{Ex}(\mathscr{A},\mathbf{Ab})$ given by $F \mapsto F \circ E$.

Given a natural transformation $\tau : E \to E'$ where $E, E' : \mathscr{A} \to \mathscr{A}'$ are 1-morphisms in $\mathbb{ABEX}^{\otimes}$, we define the natural transformation $\xi(\tau) : E^* \to E'^*$ to have component at $F \in \mathrm{Ex}(\mathscr{A}',\mathbf{Ab})$

$$\xi(\tau)_F : E^*(F) \to E'^*(F)$$

the natural transformation with component at $A \in \mathscr{A}$ given by

$$F(E(A)) \xrightarrow{F(\tau_A)} F(E'(A)).$$

Proof. By Theorem 3.4.2, $\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ is an fp-hom-closed definable subcategory of $\mathscr{A}$-Mod and S_{Ex} is a tensor-ideal of $(\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. Therefore we can define a monoidal structure on $\mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ (as in Definition 3.3.9) such that the localisation functor $q : (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \to (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Ex}} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ is a monoidal functor.

Note that the functor $\mathcal{Y}^2 : \mathscr{A} \to (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ given by $A \mapsto ((A, -), -)$ is monoidal with respect to Day convolution product and the equivalence $\mathscr{A} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ from [51, Theorem 2.2] can be taken to be $q \circ \mathcal{Y}^2$. Therefore, this equivalence is monoidal meaning the monoidal structure on $\mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ is exact. In turn this implies, by Proposition 3.3.10, that $\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ satisfies the exactness criterion. Therefore ξ is well defined on objects.

Next we need to show that, given a morphism $E : \mathscr{A} \to \mathscr{B}$ in $\mathbb{ABEX}^{\otimes}$, $E^* : \mathrm{Ex}(\mathscr{B}, \mathbf{Ab}) \to \mathrm{Ex}(\mathscr{A}, \mathbf{Ab})$ given by $F \mapsto F \circ E$ is a morphism in $\mathbb{DEF}^{\otimes}$ that is $(E^*)_0 : \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})) \to \mathrm{fun}(\mathrm{Ex}(\mathscr{B}, \mathbf{Ab}))$ is monoidal.

By the original anti-equivalence in [51], we have the following commutative diagram.

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\cong"] \arrow[d, "E"'] & \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})) \arrow[d, "(E^*)_0"] \\
\mathscr{B} \arrow[r, "\cong"] & \mathrm{fun}(\mathrm{Ex}(\mathscr{B}, \mathbf{Ab}))
\end{tikzcd}$$

We have shown above that the equivalence given by the horizontal maps is monoidal. Therefore the inverse equivalence $\mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})) \to \mathscr{A}$ is also monoidal and $(E^*)_0$ is naturally isomorphic to a monoidal functor, hence monoidal.

Finally, ξ acts on natural transformations in the same way as the original anti-equivalence, (forgetting the monoidal structure) and therefore is a well-defined 2-functor. $\square$

3.5 Completing the proof of Theorem 3.2.1

Recall Theorem 3.2.1 below.

Theorem 3.2.1. *There exists a 2-category anti-equivalence between $\mathbb{ABEX}^{\otimes}$ and $\mathbb{DEF}^{\otimes}$ given on objects by $\mathscr{A} \mapsto (\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}), \mathscr{A}\text{-Mod}, \otimes)$ where the monoidal structure, $\otimes$, on $\mathscr{A}$-Mod is induced by the monoidal structure on $\mathscr{A}$ via Day convolution product. Conversely, the anti-equivalence maps an object $(\mathcal{D}, \mathcal{C}, \otimes)$ in $\mathbb{DEF}^{\otimes}$ to the skeletally small abelian category $\mathrm{fun}(\mathcal{D}) = (\mathcal{D}, \mathbf{Ab})^{\Pi\rightarrow}$ with monoidal structure induced by Day convolution product on $\mathcal{C}^{\mathrm{fp}}$-mod (see Definition 3.3.9).*

It remains to prove that the 2-functors $\theta : (\mathbb{DEF}^{\otimes})^{\mathrm{op}} \to \mathbb{ABEX}^{\otimes}$ from Theorem 3.3.12 and $\xi : (\mathbb{ABEX}^{\otimes})^{\mathrm{op}} \to \mathbb{DEF}^{\otimes}$ from Theorem 3.4.4 give an anti-equivalence between $\mathbb{ABEX}^{\otimes}$ and $\mathbb{DEF}^{\otimes}$. By the anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$ in [51], we know that there exist equivalences $\epsilon_{\mathscr{A}} : \mathscr{A} \to \theta(\xi(\mathscr{A})) = \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ for every $\mathscr{A} \in \mathbb{ABEX}^{\otimes}$ and $\epsilon_{\mathcal{D}} : \mathcal{D} \to \xi(\theta(\mathcal{D})) = \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab})$ for every $(\mathcal{D}, \mathcal{C}, \otimes) \in \mathbb{DEF}^{\otimes}$. It remains to prove that these equivalences are morphisms in $\mathbb{ABEX}^{\otimes}$ and $\mathbb{DEF}^{\otimes}$ respectively.

Proposition 3.5.1. *For any $\mathscr{A} \in \mathbb{ABEX}^{\otimes}$ the functor $\epsilon_{\mathscr{A}} : \mathscr{A} \to \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ given by $\epsilon_{\mathscr{A}}(A) = \mathrm{ev}_A$ is monoidal. Here $\mathrm{ev}_A : \mathrm{Ex}(\mathscr{A}, \mathbf{Ab}) \to \mathbf{Ab}$ maps an exact functor $F : \mathscr{A} \to \mathbf{Ab}$ to $F(A)$.*

Similarly, for any $(\mathcal{D}, \mathcal{C}, \otimes) \in \mathbb{DEF}^{\otimes}$ the functor $\epsilon_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab})$ given by $\epsilon_{\mathcal{D}}(X) = \mathrm{ev}_X : \mathrm{fun}(\mathcal{D}) \to \mathrm{Ab}$ where ev_X denotes the functor given by 'evaluation at X' (as in the proof of Theorem 2.3 in [51]) is a morphism in $\mathbb{DEF}^{\otimes}$.

Proof. By [49, Lemma 12.9 and Theorem 12.10] the functor

$$(\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \xrightarrow{q} (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Ex}} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})),$$

maps a finitely presented functor $F : \mathscr{A}\text{-mod} \to \mathbf{Ab}$ to $\overrightarrow{F}|_{\mathcal{D}}$ that is the restriction to $\mathcal{D}$ of the unique direct limit extension of F. By the Yoneda lemma, $\epsilon_{\mathscr{A}} : \mathscr{A} \to \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ is naturally equivalent to the functor

$$\mathscr{A} \xrightarrow{\mathcal{Y}^2} (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \xrightarrow{q} (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Ex}} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab})),$$

where $\mathcal{Y}^2 : \mathscr{A} \to (\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ denotes the Yoneda embedding $A \mapsto ((A, -), -)$. Therefore, as the Yoneda embedding is monoidal with respect to Day convolution product and the monoidal structure on $\mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ is defined such that the localisation functor q and the equivalence $(\mathscr{A}\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}/\mathsf{S}_{\mathrm{Ex}} \simeq \mathrm{fun}(\mathrm{Ex}(\mathscr{A}, \mathbf{Ab}))$ are monoidal, $\epsilon_{\mathscr{A}}$ is a monoidal functor.

Next we show that, for all $(\mathcal{D}, \mathcal{C}, \otimes)$ in $\mathbb{DEF}^{\otimes}$, $(\epsilon_{\mathcal{D}})_0 : \mathrm{fun}(\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab})) \to \mathrm{fun}(\mathcal{D})$ is monoidal. By [51], $\epsilon_{\mathrm{fun}(\mathcal{D})} : \mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab}))$ is an equivalence so we have a functor, $\gamma : \mathrm{fun}(\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab})) \xrightarrow{\sim} \mathrm{fun}(\mathcal{D})$, which is both right and left adjoint to $\epsilon_{\mathrm{fun}(\mathcal{D})}$. We show that $(\epsilon_{\mathcal{D}})_0$ is naturally isomorphic to γ.

The unit of the adjunction $\gamma \dashv \epsilon_{\mathrm{fun}(\mathcal{D})}$ gives a natural isomorphism $\eta : \mathrm{Id}_{\mathrm{fun}(\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab}))} \xrightarrow{\sim} \epsilon_{\mathrm{fun}(\mathcal{D})} \circ \gamma$.

Now, for $X \in \mathcal{D}$ and $F \in \mathrm{fun}(\mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab}))$, $(\epsilon_{\mathcal{D}})_0((\epsilon_{\mathrm{fun}(\mathcal{D})} \circ \gamma)(F))(X) = \mathrm{ev}_{(\gamma)(F)}(\mathrm{ev}_X) = \mathrm{ev}_X(\gamma(F)) = \gamma(F)(X)$, so $(\epsilon_{\mathcal{D}})_0 \circ \epsilon_{\mathrm{fun}(\mathcal{D})} \circ \gamma = \gamma$. Therefore the composition of the natural isomorphism η and the functor $(\epsilon_{\mathcal{D}})_0$ gives a natural isomorphism

$$(\epsilon_{\mathcal{D}})_0\eta : (\epsilon_{\mathcal{D}})_0 \to (\epsilon_{\mathcal{D}})_0 \circ \epsilon_{\mathrm{fun}(\mathcal{D})} \circ \gamma = \gamma.$$

We have already seen that $\epsilon_{\mathrm{fun}(\mathcal{D})}$ is monoidal and therefore we can take γ to also be monoidal (e.g. see [24, Remark 1.5.3]).

Therefore $(\epsilon_{\mathcal{D}})_0$ is naturally isomorphic to a monoidal functor and so is itself a monoidal functor. Hence $\epsilon_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ex}(\mathrm{fun}(\mathcal{D}, \mathbf{Ab}))$ is a morphism in $\mathbb{DEF}^{\otimes}$ as required. □

Remark 3.5.2. The following diagram commutes, where the 2-functors denoted by $\mathscr{F}$ are the forgetful 2-functors and the vertical maps are the 2-category anti-equivalences.

$$\begin{tikzcd}
\mathbb{ABEX}^{\otimes} \arrow[r, "\mathscr{F}"] \arrow[d, leftrightarrow, "\text{Theorem 3.2.1}"'] & \mathbb{ABEX} \arrow[d, leftrightarrow, "{[51]}"] \\
\mathbb{DEF}^{\otimes} \arrow[r, "\mathscr{F}"] & \mathbb{DEF}
\end{tikzcd}$$

Chapter 4

Removing the exactness criterion

The content in this chapter is from [59].

As noted in Remark 3.4.3, for our 2-category anti-equivalence to hold, we required the monoidal structure on the skeletally small abelian category to be exact in each variable. However, given any fp-hom-closed definable subcategory $\mathcal{D}$ of a finitely accessible category $\mathcal{C}$, which satisfies Assumption 3.3.1, we can induce a right exact monoidal structure on $\text{fun}(\mathcal{D})$ as in Definition 3.3.9. In many cases, this monoidal structure on the functor category is not left exact. In this section we consider what can be said about the relationship between definability and the monoidal structure for fixed $\mathcal{C}$, where we remove the need for the exactness assumption.

4.1 The Ziegler spectrum

In this subsection we define a coarser topology, $\text{Zg}^{\text{hom}}(\mathcal{C})$, on $\text{pinj}_{\mathcal{C}}$ such that the identity morphism $\text{Zg}(\mathcal{C}) \to \text{Zg}^{\text{hom}}(\mathcal{C})$ is a continuous map.

Theorem 4.1.1. *Setting the closed subsets of* $\text{pinj}_{\mathcal{C}}$ *to be those given by the indecomposable pure-injectives contained in an fp-hom-closed definable subcategory of* $\mathcal{C}$ *defines a topology on* $\text{pinj}_{\mathcal{C}}$ *which we will call the* ***fp-hom-closed Ziegler topology*** *and denote by* $\text{Zg}^{\text{hom}}(\mathcal{C})$.

Proof. We must show that a finite union and arbitrary intersection of closed subcategories is closed. Abusing notation slightly, we will write $\mathcal{D} \cap \mathrm{pinj}_{\mathcal{C}}$ for the isomorphism classes of indecomposable pure-injective objects contained in $\mathcal{D}$, that is the closed subset of the Ziegler spectrum corresponding to $\mathcal{D}$.

We know (since the Ziegler spectrum defines a topology, e.g. [49, Theorem 14.1]) that given two definable subcategories $\mathcal{D}$ and $\mathcal{D}'$, the definable subcategory generated by their union, $\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}}$, satisfies

$$\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}} \cap \mathrm{pinj}_{\mathcal{C}} = (\mathcal{D} \cap \mathrm{pinj}_{\mathcal{C}}) \cup (\mathcal{D}' \cap \mathrm{pinj}_{\mathcal{C}}).$$

We must show that, if $\mathcal{D}$ and $\mathcal{D}'$ are fp-hom-closed, then so is $\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}}$. Notice that the Serre subcategory corresponding to $\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}}$ is given by the intersection of the Serre subcategories corresponding to $\mathcal{D}$ and $\mathcal{D}'$, say $\mathsf{S}_{\mathcal{D}}$ and $\mathsf{S}_{\mathcal{D}'}$ respectively. By Theorem 3.3.6, $\mathsf{S}_{\mathcal{D}}$ and $\mathsf{S}_{\mathcal{D}'}$ are tensor-ideals so $\mathsf{S}_{\mathcal{D}} \cap \mathsf{S}_{\mathcal{D}'}$ must also be a tensor-ideal. Applying Theorem 3.3.6 again gives that $\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}}$ is fp-hom-closed. It is straightforward to see that the intersection of fp-hom-closed definable subcategories is fp-hom-closed and this completes the proof. □

Thus we have the following tensor-analogue of Theorem 2.4.33.

Corollary 4.1.2. *Let $\mathcal{C}$ be as in Assumption 3.3.1. The correspondences in Theorem 2.4.33 restrict to bijections between:*

(i) the fp-hom-closed definable subcategories of $\mathcal{C}$,

(ii) the Serre tensor-ideals of $\mathcal{C}^{\mathrm{fp}}$-mod,

(iii) the closed subsets of $\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C})$.

In particular, the lattice of Serre tensor-ideals of $\mathcal{C}^{\mathrm{fp}}$-mod forms a spatial frame isomorphic to the lattice of open subsets $\mathbb{O}(\mathrm{Zg}^{\mathrm{hom}}(\mathcal{C}))$.

4.2 Tensor product of R-modules

Let us consider the case where $\mathcal{C} = R$-Mod for a commutative ring R. Here, R-Mod has a closed symmetric monoidal structure with tensor product given by

$\otimes_R$. The tensor unit is R and the internal hom-functor is given by the usual hom-set with R-module structure given by $(rf)(x) = rf(x) = f(rx)$ for all $x \in X$ where $f \in \hom(X, Y) = \mathrm{Hom}_R(X, Y)$ and $r \in R$. Note that R-mod is a monoidal subcategory.

The next result shows that if a functor $F \in (R\text{-mod}, \mathbf{Ab})^{fp}$ belongs to some Serre subcategory S, and if F is 'simple enough' then $G \otimes_R F \in \mathsf{S}$ for any finitely presented functor G.

Definition 4.2.1. Let $\mathcal{A}$ be a small preadditive category. The **projective dimension** of a module $M \in \mathcal{A}$-Mod is the smallest integer $n \geq 0$ such that M admits a projective resolution

$$... \to P_{n+1} \to P_n \to ... \to P_0 \to M \to 0,$$

where all $P_i = 0$ for all $i > n$. If no such integer exists, M is said to have infinite projective dimension. We denote the projective dimension of M by $\mathrm{pdim}(M)$.

Proposition 4.2.2. *Let R be a commutative ring, $\mathsf{S} \subseteq (R\text{-mod}, \mathbf{Ab})^{fp}$ be a Serre subcategory and $F \in \mathsf{S}$ satisfy* $\mathrm{pdim}(F) = 0$ *or* $\mathrm{pdim}(F) = 1$. *Then for any $G \in (R\text{-mod}, \mathbf{Ab})^{fp}$, $G \otimes F \in \mathsf{S}$, where $\otimes$ denotes the tensor product induced by $\otimes_R$ on* R-Mod.

Proof. By Lemma 3.3.5, we can take $G = (C, -)$. Throughout, let $\mathcal{D}$ be the definable subcategory associated to S as in Theorem 2.4.33.

Suppose $F \in \mathsf{S}$ satisfies $\mathrm{pdim}(F) = 0$. Then $F = (A, -)$ for some $A \in R$-mod. Therefore, for all $D \in \mathcal{D}$, $(A, D) = 0$. For any $C \in R$-mod we have $(C, -) \otimes (A, -) = (C \otimes_R A, -)$. We want to show that for all $D \in \mathcal{D}$, $(C \otimes_R A, D) = 0$. But by the adjunction isomorphism we have $(C \otimes_R A, D) \cong (C, \hom(A, D)) = (C, (A, D)) = (C, 0) = 0$, so $(C, -) \otimes (A, -) \in \mathsf{S}$, as required.

Now suppose $F \in \mathsf{S}$ satisfies $\mathrm{pdim}(F) = 1$. Then we have an exact sequence

$$0 \to (B, -) \xrightarrow{(f,-)} (A, -) \xrightarrow{\pi} F \to 0,$$

where the map $f : A \to B$ is an epimorphism in R-mod. We want to show that for all $C \in R$-mod, $(C,-) \otimes F \in \mathsf{S}$ i.e. the map $(C \otimes_R B, D) \xrightarrow{(C \otimes_R f, D)} (C \otimes_R A, D)$ is an epimorphism for all $D \in \mathcal{D}$.

As $F \in \mathsf{S}$, $(B,D) \xrightarrow{(f,D)} (A,D)$ is an isomorphism, for every $D \in \mathcal{D}$. Therefore for any $C \in R$-mod, the map $(C,(B,D)) \xrightarrow{(C,(f,D))} (C,(A,D))$ is an isomorphism and the tensor-hom adjunction gives the following commutative diagram.

$$\begin{tikzcd}
(C \otimes_R B, D) \arrow[r, "(C \otimes_R f, D)"] \arrow[d, "\cong"'] & (C \otimes_R A, D) \arrow[d, "\cong"] \\
(C,(B,D)) \arrow[r, "\cong"] & (C,(A,D))
\end{tikzcd}$$

Therefore for any $C \in R$-mod, the map $(C \otimes_R B, D) \xrightarrow{(C \otimes_R f, D)} (C \otimes_R A, D)$ is an epimorphism for all $D \in \mathcal{D}$ and $(C,-) \otimes F \in \mathsf{S}$ as required. $\square$

Remark 4.2.3. Proposition 4.2.2 does not hold for $\mathrm{pdim}(F) = 2$. Indeed, the Serre subcategory generated by T in Example 4.2.4 given below provides a counter example.

The following example is from [50, Section 13].

Example 4.2.4. *[50, Section 13] Let $R = k[\epsilon : \epsilon^2 = 0]$, where k is any field. We can define a monoidal structure on the category R-Mod with $\otimes : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ given by the usual tensor product of R-modules, $\otimes = \otimes_R$. We extend this to a monoidal structure on $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ using Day convolution product. First note that the only indecomposable R-modules are $_RR$ and $U = R/\mathrm{rad}(R) = R/\langle \epsilon \rangle \cong k$. In fact every R-module is isomorphic to a direct sum of copies of these indecomposable modules. We have $R \otimes_R R \cong R$, $R \otimes_R U \cong U$ and $U \otimes_R U \cong U$.*

Consider the exact sequence $0 \to \langle \epsilon \rangle \xrightarrow{j} R \xrightarrow{p} U \to 0$. Let S and T be determined by the exact sequences $0 \to (U,-) \xrightarrow{(p,-)} (R,-) \to S \to 0$ and $0 \to (U,-) \xrightarrow{(p,-)} (R,-) \xrightarrow{(j,-)} (\langle \epsilon \rangle, -) \to T \to 0$ in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$. By Section 13 of [50], the indecomposable functors in $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ are

$$S : M \mapsto \epsilon M,$$

$$T : M \mapsto \mathrm{ann}_M(\epsilon)/\epsilon M,$$

$$(U, -) : M \mapsto \mathrm{ann}_M(\epsilon),$$

$$W : M \mapsto M/\epsilon M,$$

and

$$(R, -) : M \mapsto M.$$

The table below shows the action of the tensor product on $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ *(given in [50, Section 13.3]).*

$\otimes$	S	T	$(U,-)$	W	$(R,-)$
S	S	0	0	S	S
T	0	$(U,-)$	$(U,-)$	T	T
$(U,-)$	0	$(U,-)$	$(U,-)$	$(U,-)$	$(U,-)$
W	S	T	$(U,-)$	W	W
$(R,-)$	S	T	$(U,-)$	W	$(R,-)$

Let us identify the definable subcategories of R-Mod *for* $R = k[\epsilon : \epsilon^2 = 0]$. *Recall that a module* $M \in R$-Mod *has form* $M = R^{(\kappa)} \oplus U^{(\lambda)}$ *for some cardinals* κ *and* λ, *(see [47, Section 6.8]). If both* κ *and* λ *are non-zero, then* $\langle M \rangle = R$-Mod. *Therefore, the only non-trivial proper definable subcategories of* R-Mod *are* $\langle R \rangle = \{R^{(\kappa)} : \kappa \text{ a cardinal}\}$ *and* $\langle U \rangle = \{U^{(\lambda)} : \lambda \text{ a cardinal}\}$.

Since $\otimes_R$ *commutes with direct sums it is easy to see that both* $\langle R \rangle^{\mathrm{def}}$ *and* $\langle U \rangle^{\mathrm{def}}$

are closed under tensor product and $\langle U\rangle^{\mathrm{def}}$ *is a tensor-ideal in* R-Mod*. Furthermore, we have* $\mathrm{hom}(R,-)=\mathrm{Hom}_R(R,-)\cong\mathrm{Id}_{R\text{-Mod}}$ *and therefore* $\mathrm{hom}(R,U)\cong U$*. It can also be checked that* $\mathrm{hom}(U,U)\cong U$*. Any object in* $\langle U\rangle^{\mathrm{def}}$ *can be written as a direct limit of finite powers of* U *and for any* $N\in R$-mod*,* $\mathrm{hom}(N,-)$ *commutes with direct limits. Therefore, since* $\mathrm{hom}(-,-)$ *commutes with finite direct sums in both variables,* $\mathrm{hom}(R,U)\cong U$ *and* $\mathrm{hom}(U,U)\cong U$ *is enough to imply that* $\langle U\rangle^{\mathrm{def}}$ *is fp-hom-closed. On the other hand,* $\mathrm{hom}(U,R)\cong U$ *meaning* $\langle R\rangle^{\mathrm{def}}$ *is not fp-hom-closed.*

Next let us consider the corresponding Serre subcategories. First take $\mathcal{D}=\langle U\rangle^{\mathrm{def}}$*. Then*

$$S(U)=\epsilon U=0,$$

$$T(U)=\mathrm{ann}_U(\epsilon)/\epsilon U=U/0\cong U,$$

$$(U,-)(U)=\mathrm{ann}_U(\epsilon)=U,$$

$$W(U)=U/\epsilon U=U/0\cong U$$

and

$$(R,-)(U)=U.$$

Therefore $\mathsf{S}_{\mathcal{D}}$ *is generated by the indecomposable functor* S *and indeed consists just of direct sums of copies of* S*. As* $\mathrm{pdim}(S)=1$*, by Proposition 4.2.2,* $G\otimes S\in S_{\mathcal{D}}$ *for every finitely presented* $G:R\text{-mod}\to\mathbf{Ab}$*. Therefore,* $\mathsf{S}_{\mathcal{D}}$ *is a Serre tensor-ideal.*

Now take $\mathcal{D}=\langle R\rangle^{\mathrm{def}}$*. Then*

$$S(R)=\epsilon R=U,$$

$$T(R)=\mathrm{ann}_R(\epsilon)/\epsilon R=U/U\cong 0,$$

$$(U,-)(R) = \mathrm{ann}_R(\epsilon) = U,$$

$$W(R) = R/\epsilon R = U$$

and

$$(R,-)(R) = R.$$

Therefore $\mathsf{S}_{\mathcal{D}}$ *is generated by the indecomposable functor* T*. As* $T \otimes T \cong (U,-)$ *this Serre subcategory is not closed under tensor product.*

In summary we get the following table, where $\langle\ \rangle^{\mathrm{def}}$ *denotes 'the definable subcategory generated by' and* $\langle\ \rangle^{\mathsf{S}}$ *denotes 'the Serre subcategory generated by'.*

Definable subcat.	Monoidal subcat.	fp-hom-closed	Tensor-ideal	Serre subcat.	Monoidal subcat.	Tensor-ideal
0	Yes	Yes	Yes	$(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$	Yes	Yes
$\langle U\rangle^{\mathrm{def}}$	Yes	Yes	Yes	$\langle S\rangle^{\mathsf{S}}$	Yes	Yes
$\langle R\rangle^{\mathrm{def}}$	Yes	No	No	$\langle T\rangle^{\mathsf{S}}$	No	No
R-Mod	Yes	Yes	Yes	0	Yes	Yes

So the Ziegler topology $\mathrm{Zg}(R\text{-Mod})$ *has underlying set* $\{[R],[U]\}$ *with the discrete topology, whereas* $\mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})$ *has closed subsets* $\emptyset$, $\{[U]\}$ *and* $\{[R],[U]\}$*.*

4.2.1 Von Neumann regular rings

Let us consider the example of von Neumann regular rings.

Definition 4.2.5. A ring R is **von Neumann regular** if for every $x \in R$ there exists some $y \in R$ such that $x = xyx$.

Proposition 4.2.6. *Let* R *be a commutative von Neumann regular ring, so the* *normal tensor product of rings, $\otimes_R$, is a symmetric closed monoidal structure on R-Mod. Every definable subcategory of R-Mod is fp-hom-closed.*

Proof. The global dimension of $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ is zero if and only if R is von Neumann regular (e.g. see [48, Proposition 10.2.20]). Thus by Lemma 4.2.2, for R von Neumann regular, every Serre subcategory of $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ is a tensor-ideal and therefore, by Theorem 3.3.6, every definable subcategory is fp-hom-closed. $\square$

Proposition 4.2.7. *Let R be a commutative von Neumann regular ring, so the normal tensor product of rings, $\otimes_R$, is a symmetric closed monoidal structure on R-Mod. Every fp-hom-closed definable subcategory $\mathcal{D}$ of R-Mod satisfies the exactness criterion.*

Proof. R is von Neumann regular if and only if every (left) R-module is flat, that is for every $M \in R$-Mod, $M \otimes_R - : R\text{-Mod} \to \mathbf{Ab}$ is exact (e.g. see [48, Theorem 2.3.22]). Therefore, since R is commutative, we obtain a symmetric closed monoidal product on R-Mod which is exact in each variable. Furthermore, by [48, Proposition 10.2.38] we have $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \simeq (R\text{-mod})^{\mathrm{op}} \simeq R\text{-mod}$ where the direction $R\text{-mod} \to (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ is given by the Yoneda embedding. Therefore, this equivalence is monoidal with respect to Day convolution product. In other words, letting $\mathcal{D} = R$-Mod, $\mathrm{fun}(\mathcal{D}) = (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \simeq R$-mod has an additive symmetric monoidal structure which is exact in each variable and coincides with the monoidal structure defined in Definition 3.3.9. Thus, by Proposition 3.3.11, $\mathcal{D} = R$-Mod satisfies the exactness criterion. Consequently any fp-hom-closed definable subcategory of R-Mod also satisfies the exactness criterion. $\square$

Remark 4.2.8. By Proposition 4.2.1, for R commutative and von Neumann regular, $\mathrm{Zg}(R\text{-Mod})$ and $\mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})$ are the same topology. Furthermore, by Proposition 4.2.7 and Proposition 3.3.10, for every definable subcategory $\mathcal{D} \subseteq R$-Mod, we can induce an exact, additive, closed, symmetric monoidal structure on the corresponding functor category $\mathrm{fun}(\mathcal{D})$.

By [48, Proposition 3.4.30] the definable subcategory generated by $M \in R$-Mod is given by $(R/\mathrm{ann}_R(M))$-Mod viewed as a full subcategory of R-Mod via $R \to$

$R/\mathrm{ann}_R(M)$. Therefore, it is easy to see directly that $\langle M \rangle^{\mathrm{def}}$ is fp-hom-closed. Futhermore, the associated Serre subcategory of $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}} \simeq R\text{-mod}$ is given by $\{X \in R\text{-mod} : (X, M) = 0\}$.

4.2.2 Coherent rings

Now we consider the case where R is coherent.

Definition 4.2.9. A commutative ring R is **coherent** if every finitely generated ideal is finitely presented, equivalently if every finitely presented R-module is coherent in the sense of Definition 2.4.18.

We can see by the above definition that a commutative ring R is coherent if and only if the category R-Mod is locally coherent. Therefore, by Example 2.4.19, the subcategory Abs-$R \subseteq$ Mod-R of absolutely pure modules is definable. In addition, by Lemma 2.4.28 R-mod is abelian.

Recall that an object X of a locally finitely presented abelian category is absolutely pure if and only if it is fp-injective (see Proposition 2.4.17). We have the following lemma.

Lemma 4.2.10. *Let $\mathcal{C}$ be a locally finitely presented abelian category with an additive closed symmetric monoidal structure such that $\mathcal{C}^{\mathrm{fp}}$ forms a monoidal subcategory. Suppose $X \in \mathcal{C}$ and $U \in \mathcal{C}^{\mathrm{fp}}$ are such that* $\mathrm{hom}(U, X)$ *is absolutely pure. Let $f : A \to B$ be a morphism in $\mathcal{C}^{\mathrm{fp}}$. If $f : A \to B$ is a monomorphism in $\mathcal{C}$, then every morphism $h : U \otimes A \to X$ factors through $U \otimes f$.*

Proof. Via the tensor-hom adjunction there exists some $h' : U \otimes B \to X$ such that $h = h' \circ (U \otimes f)$ if and only if there exists some $\widehat{h'} : B \to \mathrm{hom}(U, X)$ such that $\widehat{h} = \widehat{h'} \circ f$ where $\widehat{h} : A \to \mathrm{hom}(U, X)$ is the morphism corresponding to h via the adjunction isomorphism $(U \otimes A, X) \cong (A, \mathrm{hom}(U, X))$. But $\mathrm{hom}(U, X)$ is absolutely pure and $f : A \to B$ is a monomorphism with $A, B \in \mathcal{C}^{\mathrm{fp}}$ so as $\mathrm{hom}(U, X)$ is fp-injective, we get $\widehat{h}$ factors via f as required. $\square$

Proposition 4.2.11. *Let R be a commutative coherent ring. Any fp-hom-closed definable subcategory $\mathcal{D}$ of* Abs-R *satisfies the exactness criterion.*

Proof. Suppose $f : A \to B$ and $g : U \to V$ are morphisms in mod-R and $X \in \mathcal{D}$. Suppose further that $h : A \otimes U \to X$ satisfies $h = h' \circ (f \otimes U) = h'' \circ (A \otimes g)$ for some $h' : B \otimes U \to X$ and $h'' : A \otimes V \to X$, that is the following diagram commutes.

$$\begin{tikzcd}
A \otimes U \arrow[r, "f \otimes U"] \arrow[d, "A \otimes g"'] & B \otimes U \arrow[d, "h'"'] \\
A \otimes V \arrow[r, "h''"] & X
\end{tikzcd}$$

As mod-R is abelian, we have exact sequences in mod-R, $0 \to A' \xrightarrow{k} A \xrightarrow{f} B$ and $U \xrightarrow{g} V \xrightarrow{c} W \to 0$ where $k : A' \to A$ is the kernel of f and $c : V \to W$ is the cokernel of g. Furthermore, since $A' \otimes - :$ mod-$R \to$ mod-R is right exact we have an exact sequence

$$A' \otimes U \xrightarrow{A' \otimes g} A' \otimes V \xrightarrow{A' \otimes c} A' \otimes W \to 0.$$

Now $h'' \circ (k \otimes V) \circ (A' \otimes g) = h'' \circ (A \otimes g) \circ (k \otimes U) = h' \circ (f \otimes U) \circ (k \otimes U) = 0$. Therefore, $h'' \circ (k \otimes V)$ factors via $A' \otimes c$, say $h'' \circ (k \otimes V) = l \circ (A' \otimes c)$ for some $l : A' \otimes W \to X$.

$$\begin{tikzcd}
A' \otimes U \arrow[r, "A' \otimes g"] & A' \otimes V \arrow[r, "A' \otimes c"] \arrow[d, "k \otimes V"'] & A' \otimes W \arrow[r] \arrow[d, "k \otimes W"] \arrow[ddl, "l"'] & 0 \\
& A \otimes V \arrow[d, "h''"'] & A \otimes W \arrow[dl, "l'"] & \\
& X & &
\end{tikzcd}$$

Note that $\mathcal{D} \subseteq$ Abs-R is fp-hom-closed so $\hom(W, X)$ is absolutely pure. In addition, $k : A' \to A$ is a monomorphism therefore applying Lemma 4.2.10, $l : A' \otimes W \to X$ factors via $k \otimes W$, say $l = l' \circ (k \otimes W)$, where $l' : A \otimes W \to X$.

We have $h'' \circ (k \otimes V) = l' \circ (A \otimes c) \circ (k \otimes V)$. Setting $r = h'' - l' \circ (A \otimes c) : A \otimes V \to X$

we have $r \circ (k \otimes V) = 0$. Let $\hat{r} : A \to \hom(V, X)$ be the morphism corresponding to $r : A \otimes V \to X$ via the adjunction isomorphism. Then $\hat{r} \circ k = 0$.

Now recall that mod-R is abelian and we have exact sequence $0 \to A' \xrightarrow{k} A \xrightarrow{f} B$. Therefore, $\mathrm{coker}(k) = \mathrm{im}(f)$. Write $f : A \to B$ as $i_f \circ \pi_f$ where $\pi_f : A \to \mathrm{im}(f)$ is the cokernel of k and $i_f : \mathrm{im}(f) \to B$ is a monomorphism. Then $\hat{r}$ factors via $\pi_f : A \to \mathrm{im}(f)$, or equivalently, r factors via $\pi_f \otimes V$, say $r = r' \circ (\pi_f \otimes V)$. Noting that $\hom(V, X)$ is absolutely pure, we may apply Lemma 4.2.10 to get that $r' = r'' \circ (i_f \otimes V)$ that is r factors via $f \otimes V$.

Finally note that $h = h'' \circ (A \otimes g) = r \circ (A \otimes g) = r'' \circ (f \otimes V) \circ (A \otimes g) = r'' \circ (f \otimes g)$. Therefore we have shown that h factors via $f \otimes g$ and the exactness criterion holds for $\mathcal{D}$. $\square$

Remarks 4.2.12. (i) Proposition 4.2.11 also holds if we replace R by any skeletally small preadditive category $\mathcal{A}$ such that $\mathcal{A}$-Mod is locally coherent.

(ii) Coherent rings are precisely those rings R for which the theory of modules in the language of R-modules has a model companion [23, Theorem 4.1 and Theorem 4.8].

4.3 A rigidity assumption

Next we move on to the context where $\mathcal{C}^{\mathrm{fp}}$ forms a rigid monoidal subcategory of $\mathcal{C}$. In this setting, we get the following corollary to Theorem 3.3.6, giving a definable tensor-ideal/Serre tensor-ideal correspondence.

Corollary 4.3.1. *Let $\mathcal{C}$ be a finitely accessible category with products and suppose that $(\mathcal{C}, \otimes, 1)$ is a closed symmetric monoidal category such that $\mathcal{C}^{\mathrm{fp}}$ is a symmetric rigid monoidal subcategory. Let S be a Serre subcategory of $\mathcal{C}^{\mathrm{fp}}$*-mod *and let $\mathcal{D}$ be the corresponding definable subcategory of $\mathcal{C}$ as in (Theorem 2.4.33).*

Then, S is a Serre tensor-ideal of $\mathcal{C}^{\mathrm{fp}}$-mod *with respect to the induced tensor product if and only if $\mathcal{D}$ is a definable tensor-ideal of $\mathcal{C}$.*

Proof. By Theorem 3.3.6 we have that S is a Serre tensor-ideal if and only if $\mathcal{D}$ is fp-hom-closed. By rigidity of $\mathcal{C}^{\mathrm{fp}}$, there exists a natural equivalence $\hom(A, -) \cong$

$A^\vee \otimes -$ for all $A \in \mathcal{C}^{\mathrm{fp}}$, therefore $\mathcal{D}$ is fp-hom-closed if and only if it is closed under tensoring with objects of $\mathcal{C}^{\mathrm{fp}}$. Suppose $X \in \mathcal{C}$ and $D \in \mathcal{D}$. As $\mathcal{C}$ is finitely accessible we can write X as a direct limit $X = \varinjlim_{i \in I} X_i$ where the X_i are finitely presented. Therefore, if $\mathcal{D}$ is closed under tensoring with objects of $\mathcal{C}^{\mathrm{fp}}$, then $X \otimes D \cong (\varinjlim_{i \in I} X_i) \otimes D \cong \varinjlim_{i \in I}(X_i \otimes D) \in \mathcal{D}$, as $- \otimes D$ commutes with direct limits and $\mathcal{D}$ is closed under direct limits. $\square$

4.3.1 Examples satisfying the rigidity condition

Example 4.3.2 below gives a class of examples where the assumptions of Corollary 4.3.1 are satisfied.

Example 4.3.2. *Let G be a finite group and k be a field. The category of left kG-modules, kG-*Mod *has a closed symmetric monoidal structure with tensor product $\otimes_k$. Furthermore, the finitely generated left kG-modules form a symmetric rigid monoidal subcategory, kG-*mod*.*

*Therefore, applying Corollary 4.3.1, the definable tensor-ideals of kG-*Mod *correspond bijectively with the Serre tensor-ideals of $(kG\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.*

In particular let us consider an example from [50, Section 13].

Example 4.3.3. *[50, Section 13] Consider $R = k[\epsilon : \epsilon^2 = 0]$ as in Example 4.2.4 but suppose further that the field k has characteristic 2. Then R is a group ring. Indeed if we set $\epsilon + 1 = g$ and let $G = \langle g : g^2 = 1 \rangle \cong C_2$, then it is easy to see that $R \cong kG$ as rings. We can define a new tensor product $\otimes : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ given by $M \otimes N = M \otimes_k N$ and where the action of R is determined by $g(M \otimes N) = gM \otimes gN$.*

Note that here the tensor unit is given by U and the tensor product satisfies $R \otimes_k R \cong R^2$. We will use the notation of Example 4.2.4. The table below shows how this tensor product extends to $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ (See Section 13.5 of [50] for details of the calculation.)

$\otimes$	S	T	$(U,-)$	W	$(R,-)$
S	W	0	S	W	$(R,-)$
T	0	T	T	0	0
$(U,-)$	S	T	$(U,-)$	W	$(R,-)$
W	W	0	W	W	$(R,-)$
$(R,-)$	$(R,-)$	0	$(R,-)$	$(R,-)$	$(R,-)^2$

We get the following definable subcategory/Serre subcategory correspondence, where, as required by Corollary 4.3.1, there is a one-to-one correspondence between the definable tensor-ideals of R-Mod *and the Serre tensor-ideals of* $(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$.

Definable subcategory	Monoidal subcategory	Tensor-ideal	Serre subcategory	Monoidal subcategory	Tensor-ideal
0	Yes	Yes	$(R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$	Yes	Yes
$\langle U \rangle^{\mathrm{def}}$	Yes	No	$\langle S \rangle^{\mathsf{S}}$	No	No
$\langle R \rangle^{\mathrm{def}}$	Yes	Yes	$\langle T \rangle^{\mathsf{S}}$	Yes	Yes
R-Mod	Yes	Yes	0	Yes	Yes

So the fp-hom-closed Ziegler topology, $\mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})$, *has underlying set* $\{[R],[U]\}$ *and closed subsets* $\emptyset$, $\{[R]\}$ *and* $\{[R],[U]\}$. *As one might expect,* $\mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})$ *is different in this example to Example 4.2.4, where* R *is the same but the monoidal structure is different.*

4.4 Elementary duality

Throughout this subsection assume $\mathcal{A}$ is a small preadditive category with an additive rigid monoidal structure. We show that elementary duality (see Theorem

2.4.8) maps fp-hom-closed definable subcategories of Mod-$\mathcal{A}$ to definable tensor-ideals of $\mathcal{A}$-Mod.

Notation 4.4.1. We will denote the monoidal structure on $\mathcal{A}$ by $\otimes$, while $\otimes_{\mathcal{A}}$ denotes the tensor product of $\mathcal{A}$-modules given in Definition 2.4.7.

Definition 4.4.2. Given a finitely presented right $\mathcal{A}$-module $M \in$ mod-$\mathcal{A}$ with presentation

$$(-, m_1) \xrightarrow{(-,m)} (-, m_2) \to M \to 0$$

where $m : m_1 \to m_2$ is a morphism in $\mathcal{A}$, define (up to isomorphism) the finitely presented left $\mathcal{A}$-module $\check{M} \in \mathcal{A}$-mod to have presentation

$$(m_1^\vee, -) \xrightarrow{(m^\vee,-)} (m_2^\vee, -) \to \check{M} \to 0,$$

where $m^\vee : m_2^\vee \to m_1^\vee$ is the dual morphism to m in $\mathcal{A}$.

Similarly, given a finitely presented left $\mathcal{A}$-module $N \in \mathcal{A}$-mod with presentation

$$(n_2, -) \xrightarrow{(n,-)} (n_1, -) \to N \to 0$$

where $n : n_1 \to n_2$ is a morphism in $\mathcal{A}$, define (up to isomorphism) the finitely presented right $\mathcal{A}$-module $\check{N} \in$ mod-$\mathcal{A}$ to have presentation

$$(-, n_2^\vee) \xrightarrow{(-,n^\vee)} (-, n_1^\vee) \to \check{N} \to 0,$$

where $n^\vee : n_2^\vee \to n_1^\vee$ is the dual morphism to n in $\mathcal{A}$.

Proposition 4.4.3. *Let $\mathcal{A}$ be a small preadditive category with an additive symmetric rigid monoidal structure and induce monoidal structures on* $\mathcal{A}$-Mod *and* Mod-$\mathcal{A}$ *via Day convolution product.*

The maps $\check{(-)} : \mathcal{A}$-mod $\leftrightarrow$ mod-$\mathcal{A}$ *give an equivalence between* $\mathcal{A}$-mod *and* mod-$\mathcal{A}$.

Proof. Fix a presentation for each $N \in \mathcal{A}$-mod. First let us show that $\check{(-)}$: $\mathcal{A}$-mod $\to$ mod-$\mathcal{A}$ is functorial. Suppose $h : N \to N'$ is a morphism in $\mathcal{A}$-mod where N and N' have presentations $(n_2, -) \xrightarrow{(n,-)} (n_1, -) \to N \to 0$ and $(n'_2, -) \xrightarrow{(n',-)} (n'_1, -) \to N' \to 0$ respectively.

By projectivity of representables we can choose $h_1 : n'_1 \to n_1$ and $h_2 : n'_2 \to n_2$ such that the following diagram commutes.

$$\begin{tikzcd}
(n_2, -) \arrow[r, "{(n, -)}"] \arrow[d, "{(h_2, -)}"'] & (n_1, -) \arrow[r] \arrow[d, "{(h_1, -)}"'] & N \arrow[r] \arrow[d, "h"] & 0 \\
(n'_2, -) \arrow[r, "{(n', -)}"] & (n'_1, -) \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

Thus, $n \circ h_1 = h_2 \circ n'$ and dualising we get $h_1^\vee \circ n^\vee = n'^\vee \circ h_2^\vee$. Therefore we have the following commutative diagram where the map $\check{h}$ is uniquely determined.

$$\begin{tikzcd}
(-, {n_2}^\vee) \arrow[r, "{(-, n^\vee)}"] \arrow[d, "{(-, h_2^\vee)}"'] & (-, {n_1}^\vee) \arrow[r] \arrow[d, "{(-, h_1^\vee)}"'] & \check{N} \arrow[r] \arrow[d, dashed, "\check{h}"] & 0 \\
(-, {n'_2}^\vee) \arrow[r, "{(-, n'^\vee)}"] & (-, {n'_1}^\vee) \arrow[r] & \check{N}' \arrow[r] & 0
\end{tikzcd}$$

It is straightforward to check that any choice of h_1 and h_2 induce the same map $\check{h}$ and functoriality of $(-)^\vee : \mathcal{A} \to \mathcal{A}$ implies functoriality of $(\check{-})$. So (given a choice of presentation for all $N \in \mathcal{A}$-mod) we have a well-defined functor, $(\check{-}) : \mathcal{A}\text{-mod} \to \text{mod-}\mathcal{A}$. Furthermore, since we have a natural isomorphism $1_\mathcal{A} \to ((-)^\vee)^\vee$, by construction, the functor $(\check{-}) : \text{mod-}\mathcal{A} \to \mathcal{A}\text{-mod}$ defined similary (fixing a presentation for each $N \in \mathcal{A}$-mod) clearly gives a quasi-inverse. $\square$

Lemma 4.4.4. *For every $L \in \text{Mod-}\mathcal{A}$, $M \in \text{mod-}\mathcal{A}$ and $N \in \mathcal{A}\text{-Mod}$, we have an isomorphism*

$$(L \otimes M) \otimes_\mathcal{A} N \cong L \otimes_\mathcal{A} (\check{M} \otimes N),$$

natural in L and N.

Proof. First let us prove that for every $a \in \mathcal{A}$, we have an isomorphism $(L \otimes (-, a)) \otimes_\mathcal{A} N \cong L \otimes_\mathcal{A} ((a^\vee, -) \otimes N)$ which is natural in N and L. Recall that $- \otimes_\mathcal{A} N$ and $- \otimes (-, a)$ are both right exact and therefore preserve direct limits. Therefore, as Mod-$\mathcal{A}$ is locally finitely presentable, it is sufficient to assume that

L is finitely presented. Suppose L has presentation $(-,l_1) \xrightarrow{(-,l)} (-,l_2) \to L \to 0$. By right exactness of $- \otimes_{\mathcal{A}} N$ and $- \otimes (-,a)$ we have an exact sequence

$$(-,l_1 \otimes a) \otimes_{\mathcal{A}} N \xrightarrow{(-,l\otimes a)\otimes_{\mathcal{A}} N} (-,l_2 \otimes a) \otimes_{\mathcal{A}} N \to (L \otimes (-,a)) \otimes_{\mathcal{A}} N \to 0.$$

By definition of $\otimes_{\mathcal{A}}$, $(-,l\otimes a)\otimes_{\mathcal{A}} N : (-,l_1\otimes a)\otimes_{\mathcal{A}} N \to (-,l_2\otimes a)\otimes_{\mathcal{A}} N$ is given by $N(l \otimes a) : N(l_1 \otimes a) \to N(l_2 \otimes a)$. Thus by the Yoneda lemma we have the following commutative diagram in **Ab**.

$$\begin{tikzcd}
(-,l_1 \otimes a) \otimes_{\mathcal{A}} N \arrow[r, "(-,l\otimes a)\otimes_{\mathcal{A}} N"] \arrow[d, "\cong"'] & (-,l_2 \otimes a) \otimes_{\mathcal{A}} N \arrow[d, "\cong"'] \\
((l_1 \otimes a,-),N) \arrow[r, "((l\otimes a,-),N)"] & ((l_2 \otimes a,-),N)
\end{tikzcd}$$

By considering Lemma 3.3.8, we see that $(a^\vee,-) \otimes - : \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}$ is right adjoint to $(a,-) \otimes - : \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}$. Thus we have the following commutative diagram where the first row of downwards arrows is given by the adjointness isomorphisms and the second row is given by the Yoneda lemma.

$$\begin{tikzcd}
((l_1 \otimes a,-),N) \arrow[r, "((l\otimes a,-),N)"] \arrow[d, "\cong"'] & ((l_2 \otimes a,-),N) \arrow[d, "\cong"'] \\
((l_1,-),(a^\vee,-)\otimes N) \arrow[r, "((l,-),(a^\vee,-)\otimes N)"] \arrow[d, "\cong"'] & ((l_2,-),(a^\vee,-)\otimes N) \arrow[d, "\cong"'] \\
((a^\vee,-)\otimes N)(l_1) \arrow[r, "((a^\vee,-)\otimes N)(l)"] & ((a^\vee,-)\otimes N)(l_2)
\end{tikzcd}$$

By the definition of $\otimes_{\mathcal{A}}$ we have $((a^\vee,-)\otimes N)(l) = (-,l) \otimes_{\mathcal{A}} ((a^\vee,-)\otimes N)$.

Furthermore, by right exactness of $- \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)$ we have an exact sequence

$$(-, l_1) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \xrightarrow{(-,l)\otimes_{\mathcal{A}}((a^\vee,-)\otimes N)} (-, l_2) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)$$
$$\to L \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \to 0.$$

Thus we have an induced isomorphism $(L \otimes (-, a)) \otimes_{\mathcal{A}} N \cong L \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)$ as shown on the commutative diagram below.

$$\begin{tikzcd}
(-, l_1 \otimes a) \otimes_{\mathcal{A}} N \arrow[r, "{(-, l \otimes a) \otimes_{\mathcal{A}} N}"] \arrow[d, "\cong"'] & (-, l_2 \otimes a) \otimes_{\mathcal{A}} N \arrow[r] \arrow[d, "\cong"'] & (L \otimes (-, a)) \otimes_{\mathcal{A}} N \arrow[r] \arrow[ddd, dashed] & 0 \\
((l_1 \otimes a, -), N) \arrow[r, "{((l \otimes a, -), N)}"] \arrow[d, "\cong"'] & ((l_2 \otimes a, -), N) \arrow[d, "\cong"'] & & \\
((l_1, -), (a^\vee, -) \otimes N) \arrow[r, "{((l, -), (a^\vee, -) \otimes N)}"] \arrow[d, "\cong"'] & ((l_2, -), (a^\vee, -) \otimes N) \arrow[d, "\cong"'] & & \\
(-, l_1) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \arrow[r, "{(-, l) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N)}"] & (-, l_2) \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \arrow[r] & L \otimes_{\mathcal{A}} ((a^\vee, -) \otimes N) \arrow[r] & 0
\end{tikzcd}$$

As each of the isomorphisms in the first and second columns are natural in $(-, l_i)$ and N, the induced isomorphism is natural in L and N. Furthermore, by properties of dual morphisms in $\mathcal{A}$ we have that for every $m : m_1 \to m_2$ in $\mathcal{A}$ the following square commutes for i=1, 2.

$$\begin{tikzcd}
((l_i \otimes m_1, -), N) \arrow[r, "{((l_i \otimes m, -), N)}"] \arrow[d, "\cong"'] & ((l_i \otimes m_2, -), N) \arrow[d, "\cong"'] \\
((l_i, -), (m_1^\vee, -) \otimes N) \arrow[r, "{((l_i, -), (m^\vee, -) \otimes N)}"] & ((l_i, -), (m_2^\vee, -) \otimes N)
\end{tikzcd}$$

Therefore the induced isomorphisms $(L \otimes (-, m_i)) \otimes_{\mathcal{A}} N \cong L \otimes_{\mathcal{A}} ((m_i^\vee, -) \otimes N)$ for $i = 1, 2$ commute with any morphism $m : m_1 \to m_2$ in $\mathcal{A}$ in the following sense.

$$\begin{tikzcd}
(L\otimes(-,m_1))\otimes_{\mathcal{A}} N \arrow[r, "(L\otimes(-,m))\otimes_{\mathcal{A}} N"] \arrow[d, "\cong"'] & (L\otimes(-,m_2))\otimes_{\mathcal{A}} N \arrow[r] \arrow[d, "\cong"'] & (L\otimes M)\otimes_{\mathcal{A}} N \arrow[r] \arrow[d, dashed, "\cong"'] & 0 \\
L\otimes_{\mathcal{A}}((m_1^\vee,-)\otimes N) \arrow[r, "L\otimes_{\mathcal{A}}((m^\vee,-)\otimes N)"] & L\otimes_{\mathcal{A}}((m_2^\vee,-)\otimes N) \arrow[r] & L\otimes_{\mathcal{A}}(\check{M}\otimes N) \arrow[r] & 0
\end{tikzcd}$$

Hence the desired isomorphism $(L\otimes M)\otimes_{\mathcal{A}} N \cong L\otimes_{\mathcal{A}}(\check{M}\otimes N)$ is determined uniquely by the commutative diagram shown above. □

Theorem 4.4.5. *Let $\mathcal{A}$ be a small preadditive category with an additive, symmetric, rigid, monoidal structure and induce monoidal structures on* $\mathcal{A}$-Mod *and* Mod-$\mathcal{A}$ *via Day convolution product.*

A definable subcategory $\mathcal{D} \subseteq$ Mod-$\mathcal{A}$ *is fp-hom-closed if and only if the dual definable subcategory $\mathcal{D}^d \subseteq$* $\mathcal{A}$-Mod *is a tensor-ideal.*

Proof. By Theorem 3.3.6, $\mathcal{D} \subseteq$ Mod-$\mathcal{A}$ is an fp-hom-closed definable subcategory if and only if the corresponding Serre subcategory $\mathsf{S} \subseteq (\text{mod-}\mathcal{A}, \text{Ab})^{\text{fp}}$ is a tensor-ideal.

By [48, Proposition 10.3.5], every functor in the dual Serre subcategory $\mathsf{S}^d \subseteq (\mathcal{A}\text{-mod}, \mathbf{Ab})^{\text{fp}}$ has the form F_f^d with copresentation

$$0 \to F_f^d \to A \otimes_{\mathcal{A}} - \xrightarrow{f\otimes_{\mathcal{A}}-} B \otimes_{\mathcal{A}} -$$

for some $F_f \in \mathsf{S}$. Therefore, $X \in \mathcal{D}^d$ if and only if for every $f : A \to B$ in $\mathcal{A}$ such that $F_f \in \mathsf{S}$, $F_f^d(X) = 0$ equivalently, $f \otimes_{\mathcal{A}} X : A \otimes_{\mathcal{A}} X \to B \otimes_{\mathcal{A}} X$ is a monomorphism.

By Lemma 3.3.5, S is a tensor-ideal if and only if S is closed under tensoring with representables, that is, for all $F_f \in \mathsf{S}$ and all $M \in$ mod-$\mathcal{A}$, $(M,-)\otimes F_f = F_{M\otimes f} \in \mathsf{S}$. Thus $\mathcal{D}$ is fp-hom-closed if and only if $\mathcal{D}^d$ satisfies the following: $X \in \mathcal{D}^d$ if and only if for every $F_f \in \mathsf{S}$, and every $M \in$ mod-$\mathcal{A}$,

$$(M\otimes f)\otimes_{\mathcal{A}} X : (M\otimes A)\otimes_{\mathcal{A}} X \to (M\otimes B)\otimes_{\mathcal{A}} X$$

is a monomorphism. But by Lemma 4.4.4,

$$(M \otimes f) \otimes_{\mathcal{A}} X : (M \otimes A) \otimes_{\mathcal{A}} X \to (M \otimes B) \otimes_{\mathcal{A}} X$$

is a monomorphism if and only if

$$f \otimes_{\mathcal{A}} (\check{M} \otimes X) : A \otimes_{\mathcal{A}} (\check{M} \otimes X) \to B \otimes_{\mathcal{A}} (\check{M} \otimes X)$$

is a monomorphism. Therefore, $\mathcal{D}$ is fp-hom-closed if and only if for every $X \in \mathcal{D}^d$ and all $M \in$ mod-$\mathcal{A}$, $\check{M} \otimes X \in \mathcal{D}^d$ or equivalently for all $N \in \mathcal{A}$-mod, $N \otimes X \in \mathcal{D}^d$ as $(\check{-})$ is an equivalence (see Proposition 4.4.3). That is, $\mathcal{D}$ is a fp-hom-closed if and only if $\mathcal{D}^d$ is closed under tensoring with finitely presented left $\mathcal{A}$-modules if and only if $\mathcal{D}^d$ is a tensor-ideal, as required. $\square$

Chapter 5

Definable subcategories of tensor triangulated categories

In the rest of the thesis we focus on the triangulated setting. In this section we consider the relationship between definable subcategories and the monoidal structure in a rigidly-compactly generated tensor triangulated category. Fix a rigidly-compactly generated tensor triangulated category $\mathcal{T}$.

5.1 $\mathcal{T}$-tensor-closed definable subcategories

Since $\mathcal{T}^c$ is a skeletally small, symmetric monoidal category, we can induce a symmetric closed monoidal structure on Mod-$\mathcal{T}^c$ via Day convolution product (see Section 2.2 and [11, Appendix A]). The following lemma shows that rigidity of $\mathcal{T}^c$ implies that representable functors in Mod-$\mathcal{T}^c$ are also rigid.

Lemma 5.1.1. *For $C \in \mathcal{T}^c$, the functor $(-, C) \otimes - : \mathrm{Mod}\text{-}\mathcal{T}^c \to \mathrm{Mod}\text{-}\mathcal{T}^c$, where for $F \in \mathrm{Mod}\text{-}\mathcal{T}^c$, $(-, C) \otimes F$ is defined by Day convolution product, is both right and left adjoint to $(-, C^\vee) \otimes -$. In particular, $(-, C) \otimes -$ is exact and commutes with products and direct limits.*

Proof. We will define the unit and counit and the adjunctions.

For each $F \in \mathrm{Mod}\text{-}\mathcal{T}^c$ define $\boldsymbol{\eta}_F : F \to (-, C^\vee) \otimes (-, C) \otimes F$ to be $(-, \eta_C) \otimes F$ where $\eta_C : 1 \to C^\vee \otimes C$ is the unit of the adjunction $(C \otimes -) \dashv (C^\vee \otimes -)$

evaluated at the tensor unit, 1. Similarly, for each $F \in \text{Mod-}\mathcal{T}^c$ define $\boldsymbol{\varepsilon}_F : (-, C) \otimes (-, C^\vee) \otimes F \to F$ to be $(-, \varepsilon_C) \otimes F$ where $\varepsilon_C : C \otimes C^\vee \to 1$ is the counit of the adjunction $(C \otimes -) \dashv (C^\vee \otimes -)$ evaluated at the tensor unit, 1.

Clearly the above define natural transformations $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$. Furthermore, the triangle identities follow easily from the identities on η_C and ε_C. Therefore $(-, C^\vee) \otimes -$ is right adjoint to $(-, C) \otimes -$.

For each $F \in \text{Mod-}\mathcal{T}^c$ define $\boldsymbol{\eta}'_F : F \to (-, C) \otimes (-, C^\vee) \otimes F$ to be $(-, \eta'_C) \otimes F$ where $\eta'_C : 1 \to C \otimes C^\vee$ is the unit of the adjunction $(C^\vee \otimes -) \dashv (C \otimes -)$ evaluated at the tensor unit, 1. Similarly, for each $F \in \text{Mod-}\mathcal{T}^c$ define $\boldsymbol{\varepsilon}'_F : (-, C^\vee) \otimes (-, C) \otimes F \to F$ to be $(-, \varepsilon'_C) \otimes F$ where $\varepsilon'_C : C^\vee \otimes C \to 1$ is the counit of the adjunction $(C^\vee \otimes -) \dashv (C \otimes -)$ evaluated at the tensor unit, 1. Again, this defines natural transformations $\boldsymbol{\eta}'$ and $\boldsymbol{\varepsilon}'$ and the triangle identities follow easily from the identities on η'_C and ε'_C. So $(-, C^\vee) \otimes -$ is left adjoint to $(-, C) \otimes -$. $\square$

Next we define a monoidal structure on (the skeleton of) $\text{Coh}(\mathcal{T})$. For representables, define $(A, -) \otimes (B, -) = (A \otimes B, -)$ for all $A,\ B \in \mathcal{T}^c$. Now suppose that the tensor product is right exact. Therefore, if $F_f,\ F_g \in \text{Coh}(\mathcal{T})$ have presentations

$$(B, -) \xrightarrow{(f,-)} (A, -) \to F_f \to 0$$

and

$$(V, -) \xrightarrow{(g,-)} (U, -) \to F_g \to 0,$$

then $F_f \otimes F_g$ has presentation

$$((B \otimes U) \oplus (A \otimes V), -) \xrightarrow{\left(\left(\begin{smallmatrix} f \otimes U \\ A \otimes g \end{smallmatrix}\right), -\right)} (A \otimes U, -) \to F_f \otimes F_g \to 0.$$

It can easily be checked that this definition is well-defined up to isomorphism. Note that since Serre subcategories are isomorphism-closed, the definition of a Serre tensor-ideal of $\text{Coh}(\mathcal{T})$ makes sense, despite only having defined the monoidal structure up to isomorphism.

Proposition 5.1.2. *[36, Lemma 7.2] There is a duality*

$$\delta : (\text{mod-}\mathcal{T}^c)^{\text{op}} \xrightarrow{\sim} \text{Coh}(\mathcal{T}),$$

given by $G \mapsto \delta G$ *where* $\delta G(X) = \text{Mod-}\mathcal{T}^c(G, H_X)$ *for any* $X \in \mathcal{T}$.

Another way of describing this functor (up to isomorphism) is as follows.

Lemma 5.1.3. *[36, proof of Lemma 7.2] If* $G_f \in \text{mod-}\mathcal{T}^c$ *has presentation*

$$(-, A) \xrightarrow{(-,f)} (-, B) \to G_f \to 0,$$

then $\delta G_f \in \text{Coh}(\mathcal{T})$ *has presentation*

$$(\Sigma A, -) \xrightarrow{(f'',-)} (C, -) \to \delta G_f \to 0,$$

where $A \xrightarrow{f} B \xrightarrow{f'} C \xrightarrow{f''} \Sigma A$ *is an exact triangle in* $\mathcal{T}^c$. *That is* $\delta G_f \cong F_{f''}$ *(see Notation 2.5.6).*

Proof. Suppose $G_f \in \text{mod-}\mathcal{T}^c$ has presentation $(-, A) \xrightarrow{(-,f)} (-, B) \xrightarrow{\pi_f} G_f \to 0$. Then for any $X \in \mathcal{T}$, $\delta G_f(X) = \text{Mod-}\mathcal{T}^c(G_f, H_X)$. Suppose $\alpha : G_f \to H_X$, then $\pi_F \circ \alpha = (-, h)$ for some $h : B \to X$ by Yoneda's lemma.

$$\begin{tikzcd}
(-, A) \arrow[r, "{(-, f)}"] & (-, B) \arrow[r, "\pi_f"] \arrow[dr, dashed, "{(-, h)}"'] & G_f \arrow[r] \arrow[d, "\alpha"] & 0 \\
& & H_X &
\end{tikzcd}$$

Furthermore $h \circ f = 0$ and given any $h' : B \to X$ such that $h' \circ f = 0$, $(-, h')$ must factor through π_f, say $(-, h') = \alpha' \circ \pi_f$ for some $\alpha' : G_f \to H_X$. Therefore as abelian groups we have $\delta G_f(X) \cong \{h : B \to X : h \circ f = 0\}$. Now consider the exact triangle $A \xrightarrow{f} B \xrightarrow{f'} C \xrightarrow{f''} \Sigma A$. A morphism $h : B \to X$ satisfies $h \circ f = 0$ if and only if h factors as $h = g \circ f'$ for some $g : C \to X$. Clearly $- \circ f'$ induces an isomorphisms $(C, X)\backslash\{g : C \to X : g = g' \circ f''\} \xrightarrow{\sim} \{h : B \to X : h = g \circ f'\}$. Therefore for all $X \in \mathcal{T}$, $\delta G_f(X) \cong (C, X)\backslash\{g : C \to X : g = g' \circ f''\} \cong F_{f''}(X)$, where $F_{f''} \in \mathrm{Coh}(\mathcal{T})$ has presentation $(\Sigma A, -) \xrightarrow{(f'',-)} (C,-) \to F_{f''} \to 0$. It is straight forward to check that these isomorphisms are natural and therefore $\delta G_f \cong F_{f''}$. $\square$

We also denote the (unique up to natural isomorphism) inverse equivalence $\mathrm{Coh}(\mathcal{T})^{\mathrm{op}} \xrightarrow{\sim} \text{mod-}\mathcal{T}^c$ by δ. The monoidal structure on $\mathrm{Coh}(\mathcal{T})$ defined above is such that the functors δ in both directions are monoidal with respect to Day convolution product on mod-$\mathcal{T}^c$. That is, for $F, G \in \mathrm{Coh}(\mathcal{T})$ we have $F \otimes G \cong \delta(\delta F \otimes \delta G)$.

As δ is an equivalence, if $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ is a Serre subcategory, $\delta\mathsf{S} \subseteq \text{mod-}\mathcal{T}^c$ given by applying δ to each functor in S, is also a Serre subcategory. Let $\mathcal{D} \subseteq \mathcal{T}$ denote the definable subcategory consisting of all $X \in \mathcal{T}$ annihilated by all functors in S. Then $G \in \delta\mathsf{S}$ if and only if $\delta G \in \mathsf{S}$ if and only if $(G, H_X) = 0$ for all $X \in \mathcal{D}$.

With Notation 2.5.6 in mind, have the following Lemma.

Lemma 5.1.4. *[4, Lemma 2.2] Suppose $A \xrightarrow{f} B \xrightarrow{f'} C \xrightarrow{f''} \Sigma A$ is a distinguished triangle in $\mathcal{T}^c$. Furthermore, suppose $\mathcal{J}$ is a cohomological ideal of morphisms in $\mathcal{T}^c$ with corresponding Serre subcategory $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ (see Theorem 2.5.11) and set $\mathsf{C} = \delta\mathsf{S} \subseteq \text{mod-}\mathcal{T}^c$. Then the following are equivalent:*

(i) $f' \in \mathcal{J}$;

(ii) $G_f \in \mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$;

(iii) $F_{f''} \in \mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$.

Proof. (ii) $\iff$ (iii) Holds by Lemma 5.1.3.

(i) $\iff$ (iii) $(f', D) = 0$ if and only if every morphism $C \to D$ factors via f'' if and only if $F_{f''}(D) = 0$. $\square$

Lemma 5.1.4 gives us a clear 'picture' of the connections explored in [36] between the homological ideals of morph($\mathcal{T}^c$), the Serre subcategories of $\mathrm{Coh}(\mathcal{T})$ and the Serre subcategories of mod-$\mathcal{T}^c$.

Proposition 5.1.6 below shows that with respect to the monoidal structures defined above, δ sends Serre tensor-ideals to Serre tensor-ideals. First we give a useful lemma.

Lemma 5.1.5. *A Serre subcategory* S *of* $\mathrm{Coh}(\mathcal{T})$ *is a tensor-ideal of* $\mathrm{Coh}(\mathcal{T})$ *if and only if for all* $F \in \mathrm{Coh}(\mathcal{T})$ *and* $A \in \mathcal{T}^c$,

$$F \otimes (A, -) \in \mathsf{S}.$$

Proof. The same argument as [12, Lemma 2.12]. $\square$

Proposition 5.1.6. *The equivalence* $\delta : (\text{mod-}\mathcal{T}^c)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Coh}(\mathcal{T})$ *maps Serre tensor-ideals of* mod-$\mathcal{T}^c$ *to Serre tensor-ideals of* $\mathrm{Coh}(\mathcal{T})$.

Proof. By Lemma 5.1.5, S is a Serre tensor-ideal of $\mathrm{Coh}(\mathcal{T})$ if and only if S is closed under tensoring with representables. But S is closed under tensoring with representables if and only if for every $F_{f''} \in \mathsf{S}$ and $C \in \mathcal{T}^c$, $F_{c\otimes f''} \in \mathcal{S}$. By Lemma 5.1.4, and noting that for all $C \in \mathcal{T}^c$, $C \otimes -$ sends exact triangles to exact triangles, this property is equivalent to saying, that for all $G_f \in \delta\mathsf{S} \subseteq \text{mod-}\mathcal{T}^c$ and for all $C \in \mathcal{T}^c$, $G_{C\otimes f} \in \delta\mathsf{S}$, equivalently, $\delta\mathsf{S}$ is closed under tensoring with representable functors. It remains to apply [12, Lemma 2.12], which gives that $\delta\mathsf{S}$ is a Serre tensor-ideal of mod-$\mathcal{T}^c$ if and only if it is closed under tensoring with representable functors. $\square$

Definition 5.1.7. We say that a definable subcategory $\mathcal{D}$ of $\mathcal{T}$ is **$\mathcal{T}^c$-tensor-closed** (respectively **$\mathcal{T}$-tensor-closed**) if for all $X \in \mathcal{T}^c$ (respectively $X \in \mathcal{T}$) and for all $Y \in \mathcal{D}$, $X \otimes Y \in \mathcal{D}$.

We say that $\mathcal{D} \subseteq \mathcal{T}$ is a **definable tensor-ideal** if $\mathcal{D}$ is a definable subcategory, $\mathcal{T}$-tensor-closed and triangulated.

We say that a cohomological ideal $\mathcal{J}$ in $\mathcal{T}^c$ is **$\mathcal{T}^c$-tensor-closed** if for every $f : A \to B$ in $\mathcal{J}$ and every $C \in \mathcal{T}^c$, $C \otimes f \in \mathcal{J}$.

The following theorem gives a tensor-analogue of 2.5.11.

Theorem 5.1.8. *Let* $\mathcal{T}$ *be a rigidly-compactly generated tensor triangulated category,* $\mathcal{D}$ *be a definable subcategory of* $\mathcal{T}$, S *be the corresponding Serre subcategory of* $\mathrm{Coh}(\mathcal{T})$, $\mathsf{C} = \delta\mathsf{S} \subseteq \text{mod-}\mathcal{T}^c$ *and* $\mathcal{J}$ *be the corresponding cohomological ideal of morphisms in* $\mathcal{T}^c$ *(see 2.5.11). The following are equivalent:*

(i) *$\mathcal{D}$ is $\mathcal{T}$-tensor-closed;*

(ii) *$\mathcal{D}$ is $\mathcal{T}^c$-tensor-closed;*

(iii) S *is a Serre tensor-ideal;*

(iv) S *is closed under tensoring with representable functors;*

(v) C *is a Serre tensor-ideal;*

(vi) C *is closed under tensoring with representable functors;*

(vii) *$\mathcal{J}$ is $\mathcal{T}^c$-tensor-closed.*

Remark 5.1.9. Recall from Section 2.5, that every pp formula in the language $\mathscr{L}(\mathcal{T})$ is equivalent to a division formula ϕ_f of the form $\exists y_B, x_A = y_B f$ for some $f : A \to B$ in $\mathcal{T}^c$. By Proposition 2.5.4, ϕ_f is equivalent to $\phi_{f'}$ for $f' : A \to B'$ if and only if there exist morphisms $k : B \to B'$ and $l : B' \to B$ such that $f = l \circ f'$ and $f' = k \circ f$. Note that this defines an equivalence relation on $\mathrm{morph}(\mathcal{T}^c)$. Thus there is a bijective correspondence between the equivalence classes of pp formulas in $\mathscr{L}(\mathcal{T})$ and the equivalence classes of $\mathrm{morph}(\mathcal{T}^c)$ with respect to the equivalence relation defined above.

Furthermore, viewing pp formulas as morphisms in $\mathcal{T}^c$, the set, $\mathcal{I}$, of pp formulas that 'define' a definable subcategory $\mathcal{D} \subseteq \mathcal{T}$, in the sense that

$$\mathcal{D} = \{X \in \mathcal{T} : \phi_f(X) = 0, \ \forall f \in \mathcal{I}\},$$

is given by the cohomological ideal

$$\mathcal{I} = \mathcal{J} = \{f \in \mathrm{morph}(\mathcal{T}^c) : (f, X) = 0, \ \forall X \in \mathcal{D}\}.$$

In order to prove Theorem 5.1.8 we first give some Lemmas.

Lemma 5.1.10. *[57, Lemma 2.12] Fix $C \in \mathcal{T}^c$ and let α denote the natural isomorphism*

$$\alpha : C^\vee \otimes - \to \mathrm{hom}(C, -),$$

of functors $\mathcal{T}^c \to \mathcal{T}^c$, with components $\alpha_A : C^\vee \otimes A \to \hom(C, A)$ given by the natural evaluation map for all $A \in \mathcal{T}^c$.

The natural isomorphism α extends to a natural isomorphism of functors $\mathcal{T} \to \mathcal{T}$. In particular, for every $C \in \mathcal{T}^c$ and $X \in \mathcal{T}$ the natural evaluation map,

$$C^\vee \otimes X \to \hom(C, X),$$

is an isomorphism.

Lemma 5.1.11. *A definable subcategory $\mathcal{D}$ of $\mathcal{T}$ is $\mathcal{T}^c$-tensor-closed if and only if it is $\mathcal{T}$-tensor-closed.*

Proof. Suppose $\mathcal{D}$ is $\mathcal{T}^c$-tensor-closed. As $\mathcal{D}$ is definable, $X \in \mathcal{D}$ if and only if $(f, X) = 0$ for all $f \in \mathcal{J}$, where $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ is the associated (by Theorem 2.5.11) cohomological ideal. Suppose $f \in \mathcal{J}$ where $f : A \to B$ and $U \in \mathcal{T}^c$. If $l : U \otimes B \to X$ then the adjunction between $U \otimes -$ and $U^\vee \otimes -$ gives a map $\hat{l} : B \to U^\vee \otimes X$. As $\mathcal{D}$ is $\mathcal{T}^c$-tensor-closed, $U^\vee \otimes X \in \mathcal{D}$ so $\hat{l} \circ f = 0$. This implies that $l \circ (U \otimes f) = 0$. Therefore, for every $U \in \mathcal{T}^c$, $(U \otimes f, X) = 0$. Consider the following commutative diagram in **Ab**.

$$\begin{tikzcd}
(U \otimes B,\ X) \arrow[r, "\cong"] \arrow[d, "(U \otimes f,\ X)"'] & (U,\ B^\vee \otimes X) \arrow[d, "(U,\ f^\vee \otimes X)"] \\
(U \otimes A,\ X) \arrow[r, "\cong"] & (U,\ A^\vee \otimes X)
\end{tikzcd}$$

From the diagram we can see that, $(U \otimes f, X) = 0$ for every $U \in \mathcal{T}^c$, if and only if $(U, f^\vee \otimes X) = 0$ for every $U \in \mathcal{T}^c$, equivalently, $f^\vee \otimes X$ is a phantom map. By [12, Proposition 2.10(a)] if $f^\vee \otimes X$ is a phantom map, then so is $f^\vee \otimes X \otimes Y$ for all $Y \in \mathcal{T}$.

We have shown that if $X \in \mathcal{D}$ and $Y \in \mathcal{T}$, then $f^\vee \otimes X \otimes Y$ is a phantom map for every $f \in \mathcal{J}$. In particular, as the tensor unit $1 \in \mathcal{T}^c$, $(1, f^\vee \otimes X \otimes Y) = 0$ for every $f \in \mathcal{J}$. But then $(f, X \otimes Y) = 0$ for every $f \in \mathcal{J}$ so $X \otimes Y \in \mathcal{D}$. $\square$

Now let us give a proof of Theorem 5.1.8.

Proof. **(i)** $\leftrightarrow$ **(ii)**: A definable subcategory $\mathcal{D}$ is $\mathcal{T}$-tensor-closed if and only if it is $\mathcal{T}^c$-tensor-closed by Lemma 5.1.11.

(iii) $\leftrightarrow$ **(iv)**: A Serre subcategory $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ is a tensor-ideal if and only if it is closed under tensoring with representable functors by Lemma 5.1.5.

(v) $\leftrightarrow$ **(vi)**: A Serre subcategory $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ is a tensor-ideal if and only if it is closed under tensoring with representable functors by [12, Lemma 2.12].

(iii) $\leftrightarrow$ **(v)**: $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ is a Serre tensor-ideal if and only if $\mathsf{C} = \delta\mathsf{S} \subseteq \text{mod-}\mathcal{T}^c$ is a Serre tensor-ideal by Proposition 5.1.6).

(iv) $\leftrightarrow$ **(vii)**: A Serre subcategory $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ is closed under tensoring with representable functors if and only if for all $F_f \in \mathsf{S}$ and all $A \in \mathcal{T}^c$, $F_{A\otimes f} \in \mathsf{S}$. But by Lemma 5.1.4 this happens if and only if $\mathcal{J}$ is $\mathcal{T}^c$-tensor-closed.

(ii) $\leftrightarrow$ **(iv)**: It remains to show that $\mathcal{D}$ is $\mathcal{T}^c$-tensor-closed if and only if S is closed under tensoring with representable functors. Let $F_f \in \mathrm{Coh}(\mathcal{T})$ and $C \in \mathcal{T}^c$. Consider the following diagram, where the vertical maps are natural isomorphisms induced by the adjunction between $C \otimes -$ and $C^\vee \otimes -$.

$$\begin{tikzcd}
(C \otimes B, -) \arrow[r, "{(C\otimes f, -)}"] \arrow[d, "\cong"] & (C \otimes A, -) \arrow[r] \arrow[d, "\cong"] & (C, -) \otimes F_f \arrow[r] \arrow[d, dashed, "\cong"] & 0 \\
(B, C^\vee \otimes -) \arrow[r, "{(f, C^\vee \otimes -)}"'] & (A, C^\vee \otimes -) \arrow[r] & F_f \circ (C^\vee \otimes -) \arrow[r] & 0
\end{tikzcd}$$

So there exists a natural isomorphism $(C, -) \otimes F_f \to F_f \circ (C^\vee \otimes -)$. Therefore for all $F \in \mathsf{S}$ and $C \in \mathcal{T}^c$, $(C, -) \otimes F \in \mathsf{S}$

$\iff$ for all $F \in \mathsf{S}$, $C \in \mathcal{T}^c$ and $D \in \mathcal{D}$, $((C, -) \otimes F)(D) = 0$

$\iff$ for all $F \in \mathsf{S}$, $C \in \mathcal{T}^c$ and $D \in \mathcal{D}$, $F(C^\vee \otimes D) = 0$

$\iff$ for all $C^\vee \in \mathcal{T}^c$ and $D \in \mathcal{D}$, $C^\vee \otimes D \in \mathcal{D}$.

Noting that $(C^\vee)^\vee \cong C$ for all $C \in \mathcal{T}^c$ (see [57, Remark 1.4]), we have that the last statement is equivalent to '$\mathcal{D}$ is closed under tensoring with compact objects'. $\square$

Let us consider an example.

Example 5.1.12. *We consider some definable subcategories of* kV_4-$\underline{\mathrm{Mod}}$. *Let* ϕ *be the pp formula* $\exists y\ x = ay$ *and* ψ *be the pp formula* $\exists y, \exists z,\ x = ay \wedge x = bz$. *Clearly,* ϕ/ψ *is a pp-pair. The definable subcategory defined by closure of* ϕ/ψ *is generated by the following indecomposable pure-injectives*

(i) $M(\emptyset)$,

(ii) $M({}^n(b^{-1}a)),\ n \in \mathbb{N}$,

(iii) $M({}^n(b^{-1}a)b^{-1}),\ n \in \mathbb{Z}^{\geq 0}$,

(iv) $M({}^\infty(b^{-1}a))$,

(v) $M({}^\infty(a^{-1}b))$,

(vi) $N({}^\infty(ab^{-1}))$,

(vii) $M(ba^{-1}, \lambda, i),\ i \in \mathbb{N} \cup \{-\infty, +\infty\},\ \lambda \in k^x$,

(viii) $M(ba^{-1}, G)$.

Similarly, we can let ϕ' *be the pp formula* $\exists y\ x = by$ *and* ψ *be as above. Then* ϕ'/ψ *is a pp-pair. The definable subcategory defined by closure of* ϕ'/ψ *is generated by the following indecomposable pure-injectives*

(i) $M(\emptyset)$,

(ii) $M({}^n(b^{-1}a)),\ n \in \mathbb{N}$,

(iii) $M({}^n(ab^{-1})a),\ n \in \mathbb{Z}^{\geq 0}$,

(iv) $M({}^\infty(b^{-1}a))$,

(v) $M({}^\infty(a^{-1}b))$,

(vi) $N({}^\infty(ba^{-1}))$,

(vii) $M(ba^{-1}, \lambda, i),\ i \in \mathbb{N} \cup \{-\infty, +\infty\},\ \lambda \in k^*$,

(viii) $M(ba^{-1}, G)$.

By using computer package 'QPA' (see [28]) in GAP (see [26]), one can see that $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a) \cong P \oplus P \oplus P \oplus M(ab^{-1})$*, where* $P \cong kV_4$ *denotes the four dimensional indecomposable projective module (see Appendix A.1). Thus neither of these definable subcategories is* $\mathcal{T}$*-tensor-closed.*

Proposition 5.1.13. *Suppose* $\mathcal{X}$ *is a collection of objects in* $\mathcal{T}$ *and* $\mathcal{D} = \langle \mathcal{X} \rangle^{\mathrm{def}}$ *is the definable subcategory generated by* $\mathcal{X}$*. Then* $\mathcal{D}$ *is* $\mathcal{T}$*-tensor-closed if and only if, for all* $X \in \mathcal{X}$ *and for all* $C \in \mathcal{T}^c$*,* $C \otimes X \in \mathcal{D}$*.*

Proof. If $\mathcal{D}$ is $\mathcal{T}$-tensor-closed, $X \in \mathcal{X}$ and $C \in \mathcal{T}^c$ then clearly $C \otimes X \in \mathcal{D}$. Conversely, suppose that for all $X \in \mathcal{X}$ and for all $C \in \mathcal{T}^c$, $C \otimes X \in \mathcal{D}$. Let $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ denote the cohomological ideal associated to $\mathcal{D}$. Recall that $f \in \mathcal{J}$ if and only if $(f, X) = 0$ for all $X \in \mathcal{X}$ if and only if $(f, X) = 0$ for all $X \in \mathcal{D}$. We show that $\mathcal{J}$ is $\mathcal{T}^c$-tensor-closed. Suppose $f \in \mathcal{J}$, $C \in \mathcal{T}^c$ and $X \in \mathcal{X}$. Then $(C \otimes f, X) \cong (f, C^\vee \otimes X)$ and $(f, C^\vee \otimes X) = 0$ since $C^\vee \in \mathcal{T}^c$ meaning $C^\vee \otimes X \in \mathcal{D}$. Therefore $C \otimes f \in \mathcal{J}$ and $\mathcal{J}$ is $\mathcal{T}^c$-tensor-closed. It remains to apply Theorem 5.1.8. $\square$

Corollary 5.1.14. *If* $\mathcal{X}$ *is any* $\mathcal{T}^c$*-tensor-closed full subcategory of* $\mathcal{T}$*, then* $\langle \mathcal{X} \rangle^{\mathrm{def}}$ *is* $\mathcal{T}$*-tensor-closed.*

Example 5.1.15. *Consider the definable subcategory* $\mathcal{D} = \langle M(a) \rangle^{\mathrm{def}}$ *of* kV_4-$\underline{\mathrm{Mod}}$*. We claim that* $\mathcal{D}$ *is a* $\mathcal{T}$*-tensor-closed definable subcategory.*

Recall that the band module $M(ba^{-1}, \lambda, n)$ *for* $\lambda \in k^\times$ *and* $n \in \mathbb{N}$ *has generators* z_1^i *and* z_2^i *for* $i = 1, ..., n$ *and relations as follows.*

$az_1^i = z_2^i,\ i = 1, ..., n.$

$az_2^i = 0,\ i = 1, ..., n.$

$$bz_1^i = \begin{cases} \lambda z_2^1 & i = 1 \\ \lambda z_2^i + z_2^{i-1} & i = 2, ..., n. \end{cases}$$

$bz_2^i = 0,\ i = 1, ..., n.$

We show that $M(a) \otimes M(ba^{-1}, \lambda, n) \cong P^{(n)}$*. Denote the generators of* $M(a)$ *by* x_0 *and* x_1 *with* $ax_0 = x_1$ *and* $ax_1 = bx_0 = bx_1 = 0$*.*

Set

$$
\begin{aligned}
y_1^j &= x_0 \otimes z_1^j + x_0 \otimes z_2^j + x_1 \otimes z_2^j,\\
y_2^j &= x_0 \otimes z_1^j + x_1 \otimes z_1^j + x_1 \otimes z_2^j,\\
y_3^j &= \begin{cases} x_0 \otimes z_1^1 + x_1 \otimes z_2^1 + (\lambda+1)x_0 \otimes z_2^1 & j=1 \\ x_0 \otimes z_1^j + x_1 \otimes z_2^j + (\lambda+1)x_0 \otimes z_2^j + x_0 \otimes z_2^{j-1} & j \geq 2 \end{cases} \quad \textit{and}\\
y_4^j &= \begin{cases} x_0 \otimes z_1^1 + x_1 \otimes z_1^1 + \lambda x_0 \otimes z_2^1 + (\lambda+1)x_1 \otimes z_2^1 & j=1 \\ x_0 \otimes z_1^j + x_1 \otimes z_1^j + \lambda x_0 \otimes z_2^j + x_0 \otimes z_2^{j-1} & \\ +(\lambda+1)x_1 \otimes z_2^j + x_1 \otimes z_2^{j-1} & j \geq 2, \end{cases}
\end{aligned}
$$

for $j = 1, ..., n$.

Note that $\alpha(x_i \otimes z_j^k) = (\alpha x_i \otimes z_j^k) + (x_i \otimes \alpha z_j^k) + (\alpha x_i \otimes \alpha z_j^k)$ for $\alpha = a$ or b. It is straightforward to check that $ay_1^j = ay_2^j = y_1^j + y_2^j$, $ay_3^j = ay_4^j = y_3^j + y_4^j$, $by_1^j = by_3^j = y_1^j + y_3^j$ and $by_2^j = by_4^j = y_2^j + y_4^j$ for $j = 1, ..., n$. Recall that the indecomposable projective P is given by $P = kV_4$ with generators $1, x, y$ and xy, where a acts as $x+1$ and b acts as $y+1$. Consequently, the generators satisfy the following relations,

$$
\begin{aligned}
&a1 = x + 1 = x^2 + x = ax\\
&ay = xy + y = x^2y + xy = axy\\
&b1 = y + 1 = y^2 + y = by \quad \text{and}\\
&bx = xy + x = xy^2 + xy = yxy + xy = bxy.
\end{aligned}
$$

Therefore $1 \mapsto y_1^j$. $x \mapsto y_2^j$, $y \mapsto y_3^j$ and $xy \mapsto y_4^j$ for $j = 1, ..., n$ defines an isomorphism between P and the submodule of $M(a) \otimes M(ba^{-1}, \lambda, n)$ generated by y_1^j, y_2^j, y_3^j and y_4^j for each $j = 1, .., n$. To show that the y_i^j for $i = 1, ..., 4$ and $j = 1, ..., n$ generate $M(a) \otimes M(ba^{-1}, \lambda, n)$, one can show that the generators $x_k \otimes z_l^m$ for $k = 1, 2$, $l = 1, 2$ and $m = 1, ..., n$ can be expressed as linear combinations of *the* y_i^j*. Indeed,*

$$x_0 \otimes z_2^j = \begin{cases} \lambda^{-1}(y_1^1 + y_3^1) & if\, j = 1 \\ \lambda^{-1}(y_1^j + y_3^j) + \lambda^{-1} x_0 \otimes z_2^{j-1} & j \geq 2, \end{cases}$$

can be seen inductively to be expressible as a linear combination of the y_i^j*. Similarly, we have*

$$x_1 \otimes z_2^j = \begin{cases} \lambda^{-1}(y_2^1 + y_4^1) + x_0 \otimes z_2^1 & j = 1 \\ \lambda^{-1}(y_2^j + y_4^j) + x_0 \otimes z_2^j + \lambda^{-1} x_0 \otimes z_2^{j-1} + \lambda^{-1} x_1 \otimes z_2^{j-1} & j \geq 2. \end{cases}$$

Given that the $x_0 \otimes z_2^j$ *can be expressed as a linear combination of the* y_i^j *so can* $x_1 \otimes z_2^j$ *for* $j = 1, ..., n$*. In addition,* $x_0 \otimes z_1^j = y_1^j + x_0 \otimes z_2^j + x_1 \otimes z_2^j$ *for* $j = 1, ..., n$ *so we can also express each* $x_0 \otimes z_1^j$ *as a linear combination of the* y_i^j*. Finally an expression for* $x_1 \otimes z_1^j = y_2^j + x_0 \otimes z_1^j + x_1 \otimes z_2^j$ *in terms of the* y_i^j *can be computed using the expressions for* $x_0 \otimes z_1^j$ *and* $x_1 \otimes z_2^j$*.*

Next we show that, for any $n \in \mathbb{N}$*,* $M(a) \otimes M({}^n(b^{-1}a)) \cong P^{(n)} \oplus M(a)$*. Denote the generators of* $M(a)$ *by* x_0 *and* x_1 *with the action of* a *sending* x_0 *to* x_1*. This can be pictured as follows.*

$$\begin{tikzcd} & x_0 \arrow[dl, "a"'] \\ x_1 & \end{tikzcd}$$

Denote the generators of $M({}^n(b^{-1}a))$ *by* $x_0, x_1, ..., x_{2n}$ *with the action of* a *and* b *as pictured below.*

$$\begin{tikzcd} x_{2n} \arrow[dr, "b"'] & & x_{2n-2} \arrow[dl, "a"] & \cdots & x_2 \arrow[dr, "b"'] & & x_0 \arrow[dl, "a"] \\ & x_{2n-1} & & \cdots & & x_1 & \end{tikzcd}$$

For $j = 0, ..., n-1$, *set*

$$z_1^j = \begin{cases} x_0 \otimes x_0 + x_1 \otimes x_0 + x_0 \otimes x_{2n} + x_1 \otimes x_{2n} & j = 0 \\ x_0 \otimes x_{2j} + x_1 \otimes x_{2j} & j \geq 1 \end{cases}$$

$$z_2^j = \begin{cases} x_0 \otimes x_0 + x_0 \otimes x_1 + x_0 \otimes x_{2n} & j = 0 \\ x_0 \otimes x_{2j} + x_0 \otimes x_{2j+1} & j \geq 1 \end{cases}$$

$$z_3^j = \begin{cases} x_0 \otimes x_0 + x_1 \otimes x_0 + x_0 \otimes x_{2n-1} + x_1 \otimes x_{2n-1} + x_0 \otimes x_{2n} + x_1 \otimes x_{2n} & j = 0 \\ x_0 \otimes x_{2j} + x_1 \otimes x_{2j} + x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} & j \geq 1 \end{cases}$$

$$z_4^j = \begin{cases} x_0 \otimes x_0 + x_0 \otimes x_1 + x_0 \otimes x_{2n} + x_0 \otimes x_{2n-1} & j = 0 \\ x_0 \otimes x_{2j} + x_0 \otimes x_{2j+1} + x_0 \otimes x_{2j-1} & j \geq 1 \end{cases}$$

$$y_1 = x_0 \otimes x_0$$

$$y_2 = x_0 \otimes x_1 + x_1 \otimes x_0 + x_1 \otimes x_1.$$

It can be checked that $az_1^j = az_2^j = z_1^j + z_2^j$, $az_3^j = az_4^j = z_3^j + z_4^j$, $bz_1^j = bz_3^j = z_1^j + z_3^j$ *and* $bz_2^j = bz_4^j = z_2^j + z_4^j$ *for* $j = 0, ..., n-1$ *and therefore the submodule generated by* z_1^j, z_2^j, z_3^j *and* z_4^j *is isomorphic to* P *for each* $j = 1, ..., n$. *In addition* $ay_1 = y_2$ *and* $by_1 = ay_2 = by_2 = 0$ *so the submodule generated by* y_1 *and* y_2 *is isomorphic to* $M(a)$.

To show that the z_i^j *for* $i = 1, ..., 4$ *and* $j = 0, ..., n-1$ *together with* y_1 *and* y_2 *generate* $M(a) \otimes M({}^n(b^{-1}a))$ *we show that each generator* $x_k \otimes x_l$ *for* $k = 0, 1$ *and* $l = 0, ..., 2n$ *can be written as a linear combination of the* z_i^j, y_1 *and* y_2 *as follows.*

$$\begin{aligned} x_0 \otimes x_{2n-1} &= z_2^0 + z_4^0 \\ x_1 \otimes x_{2j-1} &= z_1^j + z_2^j + z_3^j + z_4^j, \text{ for } j = 1, ..., n-1 \\ x_0 \otimes x_{2j-1} &= z_2^j + z_4^j \text{ for } j = 1, ..., n-1 \\ x_1 \otimes x_{2n-1} &= z_1^0 + z_2^0 + z_3^0 + z_4^0 \\ x_0 \otimes x_{2j} &= z_2^j + z_2^{j+1} + z_4^{j+1} \text{ for } j = 1, ..., n-1 \\ x_1 \otimes x_{2j} &= z_1^j + z_2^j + z_2^{j+1} + z_4^{j+1} \text{ for } j = 1, ..., n-1 \end{aligned}$$

$$
\begin{aligned}
x_0 \otimes x_0 &= y_1 \\
x_1 \otimes x_0 &= y_2 + z_1^1 + z_3^1 \\
x_0 \otimes x_{2n} &= y_1 + z_2^0 + z_2^1 + z_4^1 \\
x_1 \otimes x_{2n} &= y_2 + z_1^0 + z_2^0 + z_1^1 + z_2^1 + z_3^1 + z_4^1.
\end{aligned}
$$

Next we show that, for any $n \in \mathbb{N}$, $M(a) \otimes M({}^n(ab^{-1})a) \cong P^{(n)} \oplus M(a) \oplus M(a)$. *Denote the generators of* $M(a)$ *by* x_0 *and* x_1 *with the action of* a *sending* x_0 *to* x_1. *This can be pictured as follows.*

$$
\begin{tikzcd}
 & x_0 \arrow[dl, "a"'] \\
x_1 &
\end{tikzcd}
$$

Denote the generators of $M({}^n(ab^{-1})a)$ *by* $x_0, x_1, ..., x_{2n+1}$ *with the action of* a *and* b *as pictured below.*

$$
\begin{tikzcd}
 & x_{2n} \arrow[dl, "a"'] \arrow[dr, "b"] & & x_{2n-2} \arrow[dl, "a"'] & \cdots & x_2 \arrow[dr, "b"] & & x_0 \arrow[dl, "a"'] \\
x_{2n+1} & & x_{2n-1} & & \cdots & & x_1 &
\end{tikzcd}
$$

For $j = 1, ..., n$, *set*

$$
\begin{aligned}
z_1^j &= x_0 \otimes x_{2j} + x_1 \otimes x_{2j} \\
z_2^j &= x_0 \otimes x_{2j} + x_0 \otimes x_{2j+1} \\
z_3^j &= x_0 \otimes x_{2j} + x_1 \otimes x_{2j} + x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} \\
z_4^j &= x_0 \otimes x_{2j} + x_0 \otimes x_{2j+1} + x_0 \otimes x_{2j-1}
\end{aligned}
$$

$$\begin{aligned}
y_1 &= x_0 \otimes x_0 \\
y_2 &= x_0 \otimes x_1 + x_1 \otimes x_0 + x_1 \otimes x_1 \\
y_1' &= x_0 \otimes x_{2n+1} \\
y_2' &= x_1 \otimes x_{2n+1}
\end{aligned}$$

It can be checked that $az_1^j = az_2^j = z_1^j + z_2^j$, $az_3^j = az_4^j = z_3^j + z_4^j$, $bz_1^j = bz_3^j = z_1^j + z_3^j$ *and* $bz_2^j = bz_4^j = z_2^j + z_4^j$ *for* $j = 0, ..., n-1$ *and therefore the submodule generated by* z_1^j, z_2^j, z_3^j *and* z_4^j *is isomorphic to* P *for each* $j = 0, ..., n-1$. *In addition* $ay_1 = y_2$ *and* $by_1 = ay_2 = by_2 = 0$ *and similarly for* y_1' *and* y_2' *so the* 2*-dimensional submodules generated by* y_1 *and* y_2 *(respectively* y_1' *and* y_2'*) are isomorphic to* $M(a)$.

To show that the z_i^j *for* $i = 1, ..., 4$ *and* $j = 0, ..., n-1$, y_1,y_1', y_2 *and* y_2' *generate* $M(a) \otimes M({}^n(ab^{-1})a)$ *we show that each generator* $x_k \otimes x_l$ *for* $k = 0, 1$ *and* $l = 0, ..., 2n+1$ *can be written as a linear combination of the* z_i^j, y_1, y_1', y_2 *and* y_2' *as follows.*

$$\begin{aligned}
x_0 \otimes x_0 &= y_1 \\
x_0 \otimes x_{2j-1} &= z_2^j + z_4^j \quad j = 1, ..., n \\
x_0 \otimes x_{2n+1} &= y_1' \\
x_0 \otimes x_{2j} &= \begin{cases} z_2^j + z_2^{j+1} + z_4^{j+1} & \text{for } j = 1, ..., n-1 \\ z_2^j + y_1' & j = n \end{cases} \\
x_1 \otimes x_{2n+1} &= y_2' \\
x_1 \otimes x_{2j} &= \begin{cases} z_1^j + z_2^j + z_2^{j+1} + z_4^{j+1} & j = 1, ..., n-1 \\ z_1^j + z_2^j + y_1' & j = n \end{cases} \\
x_1 \otimes x_{2j-1} &= z_1^j + z_2^j + z_3^j + z_4^j, \ \text{for } j = 1, ..., n \\
x_1 \otimes x_0 &= y_2 + z_1^1 + z_3^1.
\end{aligned}$$

Next we show that, for any $n \in \mathbb{N}$, $M(a) \otimes M({}^n(ab^{-1})) \cong P^{(n)} \oplus M(a)$. *Denote* *the generators of $M(a)$ by x_0 and x_1 with the action of a sending x_0 to x_1. This can be pictured as follows.*

$$\begin{tikzcd}
 & x_0 \arrow[dl, "a"'] \\
x_1 &
\end{tikzcd}$$

Denote the generators of $M({}^n(ab^{-1}))$ by $x_0, x_1, ..., x_{2n}$ with the action of a and b as pictured below.

$$\begin{tikzcd}
 & x_{2n-1} \arrow[dl, "a"'] \arrow[dr, "b"] & & \cdots & & x_1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\
x_{2n} & & x_{2n-2} & \cdots & x_2 & & x_0
\end{tikzcd}$$

For $j = 1, ..., n$, set

$$\begin{aligned}
z_1^j &= x_0 \otimes x_{2j-1} + x_0 \otimes x_{2j} \\
z_2^j &= x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} \\
z_3^j &= x_0 \otimes x_{2j-1} + x_0 \otimes x_{2j} + x_0 \otimes x_{2j-2} \\
z_4^j &= x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} + x_0 \otimes x_{2j-2} + x_1 \otimes x_{2j-2} \\
y_1 &= x_0 \otimes x_{2n} \\
y_2 &= x_1 \otimes x_{2n}
\end{aligned}$$

It can be checked that $az_1^j = az_2^j = z_1^j + z_2^j$, $az_3^j = az_4^j = z_3^j + z_4^j$, $bz_1^j = bz_3^j = z_1^j + z_3^j$ and $bz_2^j = bz_4^j = z_2^j + z_4^j$ for $j = 1, ..., n$ and therefore the submodule generated by z_1^j, z_2^j, z_3^j and z_4^j is isomorphic to P for each $j = 1, ..., n$. In addition $ay_1 = y_2$ and $by_1 = ay_2 = by_2 = 0$ so the 2-dimensional submodule generated by y_1 and y_2 is isomorphic to $M(a)$.

To show that the z_i^j for $i = 1, ..., 4$ and $j = 1, ..., n$, y_1 and y_2 generate $M(a) \otimes M({}^n(ab^{-1}))$ we show that each generator $x_k \otimes x_l$ for $k = 0, 1$ and $l = 0, ..., 2n$ can be written as a linear combination of the z_i^j, y_1, and y_2 as follows.

$$
\begin{aligned}
x_0 \otimes x_{2n} &= y_1 \\
x_1 \otimes x_{2n} &= y_2 \\
x_0 \otimes x_{2j-2} &= z_1^j + z_3^j, \text{ for } j = 1, ..., n \\
x_1 \otimes x_{2j-2} &= z_1^j + z_2^j + z_3^j + z_4^j, \text{ for } j = 1, ..., n \\
x_0 \otimes x_{2j-1} &= \begin{cases} z_1^j + z_1^{j+1} + z_3^{j+1} & \text{for } j = 1, ..., n-1 \\ z_1^j + y_1 & j = n \end{cases} \\
x_1 \otimes x_{2j-1} &= \begin{cases} z_1^j + z_2^j + z_1^{j+1} + z_3^{j+1} & j = 1, ..., n-1 \\ z_1^j + z_2^j + y_1 & j = n \end{cases}
\end{aligned}
$$

Finally we show that, for any $n \in \mathbb{N}$, $M(a) \otimes M({}^n(b^{-1}a)b^{-1}) \cong P^{(n+1)}$. Denote the generators of $M(a)$ by x_0 and x_1 with the action of a as pictured below.

$$
\begin{tikzcd}
 & x_0 \arrow[dl, "a"'] \\
x_1 &
\end{tikzcd}
$$

Denote the generators of $M({}^n(b^{-1}a)b^{-1})$ by $x_0, x_1, ..., x_{2n+1}$ with the action of a and b as pictured below.

$$
\begin{tikzcd}
x_{2n+1} \arrow[dr, "b"'] & & x_{2n-1} \arrow[dl, "a"] & ... & x_3 \arrow[dr, "b"'] & & x_1 \arrow[dl, "a"] \arrow[dr, "b"] & \\
 & x_{2n} & & ... & & x_2 & & x_0
\end{tikzcd}
$$

For $j = 1, ..., n+1$, set

$$z_1^j = \begin{cases} x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} & j = 1, ..., n \\ x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} + x_0 \otimes x_0 + x_1 \otimes x_0 & j = n+1 \end{cases}$$

$$z_2^j = \begin{cases} x_0 \otimes x_{2j-1} + x_0 \otimes x_{2j} & j = 1, ..., n \\ x_0 \otimes x_{2j-1} + x_0 \otimes x_0 & j = n+1 \end{cases}$$

$$z_3^j = \begin{cases} x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} + x_0 \otimes x_{2j-2} + x_1 \otimes x_{2j-2} & j = 1, ..., n \\ x_0 \otimes x_{2j-1} + x_1 \otimes x_{2j-1} + x_0 \otimes x_{2j-2} + x_1 \otimes x_{2j-2} & \\ +x_0 \otimes x_0 + x_1 \otimes x_0 & j = n+1 \end{cases}$$

$$z_4^j = \begin{cases} x_0 \otimes x_{2j-1} + x_0 \otimes x_{2j} + x_0 \otimes x_{2j-2} & j = 1, ..., n \\ x_0 \otimes x_{2j-1} + x_0 \otimes x_0 + x_0 \otimes x_{2j-2} & j = n+1 \end{cases}$$

It can be checked that $az_1^j = az_2^j = z_1^j + z_2^j$, $az_3^j = az_4^j = z_3^j + z_4^j$, $bz_1^j = bz_3^j = z_1^j + z_3^j$ and $bz_2^j = bz_4^j = z_2^j + z_4^j$ for $j = 1, ..., n+1$ and therefore the submodule generated by z_1^j, z_2^j, z_3^j and z_4^j is isomorphic to P for each $j = 1, ..., n+1$.

To show that $M(a) \otimes M({}^n(b^{-1}a)b^{-1})$ is generated by the z_i^j for $i = 1, ..., 4$ and $j = 1, ..., n$, we show that each generator $x_k \otimes x_l$ for $k = 0, 1$ and $l = 0, ..., 2n+1$ can be written as a linear combination of the z_i^j as follows.

$$x_0 \otimes x_{2j-2} = z_2^j + z_4^j, \text{ for } j = 1, ..., n+1$$

$$x_1 \otimes x_{2j-2} = z_1^j + z_2^j + z_3^j + z_4^j, \text{ for } j = 1, ..., n+1$$

$$x_0 \otimes x_{2j-1} = \begin{cases} z_2^j + z_2^{j+1} + z_4^{j+1} & \text{for } j = 1, ..., n \\ z_2^j + z_2^1 + z_4^1 & j = n+1 \end{cases}$$

$$x_1 \otimes x_{2j-1} = \begin{cases} z_1^j + z_2^j + z_2^{j+1} + z_4^{j+1} & j = 1, ..., n \\ z_1^j + z_2^j + z_1^1 + z_2^1 + z_3^1 + z_4^1 & j = n+1 \end{cases}$$

Thus we have shown that for every indecomposable finitely presented module M in kV_4-Mod, $M(a) \otimes M$ is a direct sum of copies of $M(a)$ and P. Therefore, in kV_4-$\underline{\mathrm{Mod}}$, $M \otimes M(a) \in \langle M(a) \rangle^{\mathrm{def}}$ for every indecomposable compact object M. Since the tensor product commutes with direct sums and every compact object is a direct sum of indecomposable finitely presented modules, it follows that $M \otimes M(a) \in \langle M(a) \rangle^{\mathrm{def}}$ for all compact objects M. By Proposition 5.1.13, the definable subcategory generated by $M(a)$ is $\mathcal{T}^c$-tensor-closed. Finally applying Theorem 5.1.8 we conclude that $\langle M(a) \rangle^{\mathrm{def}}$ is $\mathcal{T}$-tensor-closed.

Notation 5.1.16. Given a subcategory $\mathcal{X}$ of a finitely accessible category $\mathcal{C}$, we denote by $\varinjlim \mathcal{X}$ the closure of $\mathcal{X}$ under direct limits.

We have established a one-to-one correspondence between the $\mathcal{T}$-tensor-closed definable subcategories of $\mathcal{T}$, the Serre tensor-ideals of $\mathrm{Coh}(\mathcal{T})$ and mod-$\mathcal{T}^c$ and the $\mathcal{T}^c$-tensor-closed cohomological ideals of morphisms of $\mathcal{T}^c$. In [12], Balmer, Krause and Stevenson associate to a Serre tensor-ideal C of mod-$\mathcal{T}^c$ a pure-injective $E \in \mathcal{T}$ such that $\varinjlim \mathsf{C} = \ker(H_E \otimes -)$ (see [12, Theorem 3.5]).

Let $\overline{H} : \mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{T}^c / \varinjlim \mathsf{C}$ denote the composition

$$\mathcal{T} \xrightarrow{H} \mathrm{Mod}\text{-}\mathcal{T}^c \xrightarrow{q} \mathrm{Mod}\text{-}\mathcal{T}^c / \varinjlim \mathsf{C}.$$

Such a pure-injective, E, is constructed by considering the injective hull, I, of $\overline{H}_1$ in Mod-$\mathcal{T}^c/\varinjlim \mathsf{C}$, where $1 \in \mathcal{T}$ is the tensor unit. Here the monoidal structure on Mod-$\mathcal{T}^c$ is induced by that of $\mathcal{T}^c$ by Day convolution product (see Section 2.2) and the monoidal structure on the localisation Mod-$\mathcal{T}^c/\varinjlim \mathsf{C}$ extends that in Definition 3.3.9. The quotient functor $q : \mathrm{Mod}\text{-}\mathcal{T}^c \to \mathrm{Mod}\text{-}\mathcal{T}^c/\varinjlim \mathsf{C}$ admits a right adjoint, r, which preserves injectives (see [12, Proposition 2.13(e)]) and the restricted Yoneda functor $H : \mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{T}^c$ identifies the pure-injective objects in $\mathcal{T}$ and the injective objects in Mod-$\mathcal{T}^c$. Therefore there exists a unique up to unique isomorphism pure-injective $E \in \mathcal{T}$, such that H_E is isomorphic in Mod-$\mathcal{T}^c$ to $r(I)$ that is the image under the right adjoint r of the injective hull in Mod-$\mathcal{T}^c/\varinjlim \mathsf{C}$ of $\overline{H}_1$.

We have the following 'picture'.

$$\begin{tikzcd} \mathsf{C} \underset{\substack{Serre\\ \otimes-ideal}}{\subseteq} \text{mod-}\mathcal{T}^c \arrow[rr, leftrightarrow, "\text{Proposition 5.1.6}"] \arrow[d, "{([12], 3.1)}"'] & & \mathsf{S} \underset{\substack{Serre\\ \otimes-ideal}}{\subseteq} \mathrm{Coh}(\mathcal{T}) \\ E \underset{\substack{pure-\\ injective}}{\in} \mathcal{T} & & \mathcal{D} \underset{\substack{\mathcal{T}-tensor-closed\\ definable}}{\subseteq} \mathcal{T} \arrow[u, leftrightarrow, "\text{Theorem 5.1.8}"'] \end{tikzcd}$$

Proposition 5.1.17. *Let $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ be a Serre tensor-ideal and $E \in \mathcal{T}$ be the associated pure-injective as in [12, Construction 3.1]. Suppose $\mathcal{D}$ is the definable subcategory corresponding to the Serre subcategory $\mathsf{S} = \delta\mathsf{C} \subseteq \mathrm{Coh}(\mathcal{T})$.*

Then the cohomological ideal associated to the definable subcategory $\mathcal{D} \subseteq \mathcal{T}$ is given by $\mathcal{J} = \{f' \in \mathrm{morph}(\mathcal{T}^c) : E \otimes f' = 0\}$.

Proof. Suppose $f' : B \to C$ in $\mathcal{T}^c$ and $E \otimes f' = 0$. Completing f' to a triangle in $\mathcal{T}^c$ and tensoring with E, we get an exact triangle

$$E \otimes A \xrightarrow{E \otimes f} E \otimes B \xrightarrow{E \otimes f'} E \otimes C \xrightarrow{E \otimes f''} E \otimes \Sigma C.$$

Since $E \otimes f' = 0$ every morphism $X \to E \otimes B$ factors via $E \otimes f$. Since $H_E \otimes G_f$ has presentation

$$(-, E \otimes A)|_{\mathcal{T}^c} \xrightarrow{(-, E \otimes f)} (-, E \otimes B)|_{\mathcal{T}^c} \to H_E \otimes G_f \to 0,$$

$H_E \otimes G_f = 0$ meaning $G_f \in \mathsf{C}$ as $\varinjlim \mathsf{C} = \ker(H_E \otimes -)$. Therefore, by Lemma 5.1.4, $f' \in \mathcal{J}$.

Conversely, suppose $g' : V \to W$ in $\mathcal{T}^c$ is in the cohomological ideal associated to $\mathcal{D}$. Complete g' to a triangle and rotate to obtain a triangle of form $U \xrightarrow{g} V \xrightarrow{g'} W \xrightarrow{g''} \Sigma U$. By Lemma 5.1.4, $G_g \in \mathsf{C}$. Therefore, $H_E \otimes G_g = 0$. Note that we have exact sequence

$$(-, E \otimes U)|_{\mathcal{T}^c} \xrightarrow{(-, E \otimes g)} (-, E \otimes V)|_{\mathcal{T}^c} \to H_E \otimes G_g \to 0.$$

Considering the exact triangle, $E \otimes U \xrightarrow{E\otimes g} E \otimes V \xrightarrow{E\otimes g'} E \otimes W \xrightarrow{E\otimes g''} E \otimes \Sigma U$, we see that $H_E \otimes G_g = 0$ implies that $E \otimes g'$ is phantom. But by [12, Corollary 3.6], $E \otimes g'$ is phantom if and only if $E \otimes g' = 0$, as $g' \in \mathrm{morph}(\mathcal{T}^c)$. □

Remark 5.1.18. In other words, Proposition 5.1.17 tells us that for any $\mathcal{T}$-tensor-closed definable subcategory $\mathcal{D} \subseteq \mathcal{T}$ there is a pure-injective E such that $\mathcal{D}$ is defined by a collection of pp formulas ϕ_f of the form $\exists y_B,\ x_A = y_B f$ where f ranges over the morphisms in $\mathcal{T}^c$ such that $E \otimes f = 0$.

Proposition 5.1.19. *Suppose we have a Serre tensor-ideal,* $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$*, and corresponding pure-injective* $E \in \mathcal{T}$ *as in [12, Construction 3.1]. Set* $\mathsf{S} = \delta\mathsf{C} \subseteq \mathrm{Coh}(\mathcal{T})$ *and let* $\mathcal{D} \subseteq \mathcal{T}$ *be the corresponding* $\mathcal{T}$*-tensor-closed definable subcategory as in Theorem 5.1.8. Then* $E \in \mathcal{D}$*.*

Proof. By Proposition 5.1.17, $E \in \mathcal{D}$ if and only if, $(f, E) = 0$ for all $f \in \mathrm{morph}(\mathcal{T}^c)$ such that $E \otimes f = 0$. Suppose $E \otimes f = 0$ where $f : A \to B$ is a morphism in $\mathcal{T}^c$ and $g : B \to E$. Then $H_{E\otimes f} = 0$ so $H_{E\otimes(g\circ f))} = 0$. By [12, Corollary 3.6] $H_{E\otimes(g\circ f))} = 0$ implies the image of $H_{g\circ f}$ under the quotient map $\text{Mod-}\mathcal{T}^c \to \text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ is zero. Therefore since E is pure-injective and $H_E \notin \varinjlim \mathsf{C}$ meaning E is in the image of the right adjoint to the quotient functor $\text{Mod-}\mathcal{T}^c \to \text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$, [12, Corollary 2.18(c)] implies that $g \circ f = 0$. □

5.2 Definable tensor-ideals

Recall that the distinction between a $\mathcal{T}$-tensor-closed definable subcategory and a definable tensor-ideal is that a definable tensor-ideal is a triangulated subcategory (Definition 5.1.7). In this section we consider the role of definable tensor-ideals in a rigidly-compactly generated tensor triangulated category $\mathcal{T}$.

Notation 5.2.1. For a full subcategory $\mathcal{X} \subseteq \mathcal{T}$ and $I \subseteq \mathbb{Z}$, we denote by ${}^{\perp_I}\mathcal{X}$ the full subcategory with objects $\{Z \in \mathcal{T} : (Z, \Sigma^i X) = 0, \forall X \in \mathcal{X}, i \in I\}$. We write just ${}^{\perp}\mathcal{X}$ for the case $I = \{0\}$.

Similarly we denote by $\mathcal{X}^{\perp_I}$ the full subcategory of $\mathcal{T}$ with objects $\{Z \in \mathcal{T} : (X, \Sigma^i Z) = 0, \forall X \in \mathcal{X}, i \in I\}$. We write $\mathcal{X}^{\perp}$ for the case $I = \{0\}$.

First we show that $\mathcal{D} \subseteq \mathcal{T}$ is a definable tensor-ideal if and only if ${}^{\perp}\mathcal{D}$ is a smashing tensor-ideal of $\mathcal{T}$. The correspondence between triangulated definable subcategories and smashing subcategories is already known (e.g. see [37] and [40, Remark 6.4]). However, for completeness we will give details of the proof before establishing a tensor version.

Definition 5.2.2. A full triangulated subcategory $\mathcal{B} \subseteq \mathcal{T}$ is said to be **smashing** if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{T}$ has a right adjoint which preserves coproducts.

Definition 5.2.3. Full subcategories $\mathcal{U}, \mathcal{V} \subseteq \mathcal{T}$ form a **torsion pair** if the following hold:

(i) $\mathcal{T}(\mathcal{U}, \mathcal{V}) = 0$;

(ii) $\mathcal{U}$ and $\mathcal{V}$ are closed under direct summands;

(iii) For every object $X \in \mathcal{T}$ there is an exact triangle

$$U \to X \to V \to \Sigma U$$

such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

A torsion pair, $(\mathcal{U}, \mathcal{V})$ is said to be a **t-structure** (respectively **co-t-structure**) if $\Sigma\mathcal{U} \subseteq \mathcal{U}$ (respectively $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$). Given a t-structure $(\mathcal{U}, \mathcal{V})$, $\mathcal{U}$ is called the **aisle** of the t-structure and $\mathcal{V}$ is called the **coaisle**. The **heart** of the t-structure $(\mathcal{U}, \mathcal{V})$ is given by $\mathcal{H}_t = \mathcal{U} \cap \Sigma\mathcal{V}$.

We will show that smashing subcategories $\mathcal{B}$ and triangulated definable subcategories $\mathcal{D}$ lie in torsion pairs $(\mathcal{B}, \mathcal{D})$. The following equivalent characterisation of a smashing subcategory will be useful.

Proposition 5.2.4. *(e.g. [46, Theorem 4.4.3]) Let $\mathcal{B} \subseteq \mathcal{T}$ be a full triangulated subcategory. Then $\mathcal{B}$ is a smashing subcategory if and only if $\mathcal{B}$ is the aisle of a t-structure $(\mathcal{B}, \mathcal{B}^{\perp})$ such that $\mathcal{B}^{\perp}$ is closed under coproducts.*

Definition 5.2.5. We say that a full subcategory $\mathcal{X} \subseteq \mathcal{T}$ is **suspended** (respectively **cosuspended**) if it is closed under direct summands, extensions and positive (respectively negative) shift.

We say that a full subcategory $\mathcal{X} \subseteq \mathcal{T}$ is **precovering** if for every $Y \in \mathcal{T}$ there exists an $X \in \mathcal{X}$ and a morphism $f : X \to Y$ such that, given any morphism $g : X' \to Y$ with $X' \in \mathcal{X}$, there exists some $h : X' \to X$ such that $g = h \circ f$. **Preenveloping** subcategories are defined dually.

Proposition 5.2.6. *([44, Proposition 1.4] and [5, Example 2.4(3)]) Let $\mathcal{V}$ be a suspended and precovering (respectively cosuspended and preenveloping) subcategory of $\mathcal{T}$. The inclusion functor $\mathcal{V} \hookrightarrow \mathcal{T}$ has a right (respectively left) adjoint.*

Proof. The proof in the suspended case follows the argument of [44, Proposition 1.4]. The cosuspended case is dual. For completeness we give the proof of the cosuspended case in full. Suppose $X \in \mathcal{T}$. To find a left adjoint $\lambda : \mathcal{T} \to \mathcal{V}$ we need to find a universal morphism $\varepsilon_X : X \to \lambda(X)$ with $\lambda(X) \in \mathcal{V}$. As $\mathcal{V}$ is preenveloping, we have a morphism $f : X \to V$ with $V \in \mathcal{V}$ such that every morphism $f' : X \to V'$ with $V' \in \mathcal{V}$ factors (not necessarily uniquely) through f. Complete f to an exact triangle, say $\Sigma^{-1}U \to X \xrightarrow{f} V \xrightarrow{\alpha} U$ and choose a $\mathcal{V}$-preenvelope of U, say $\beta : U \to W$, with $W \in \mathcal{V}$. We have the following morphism of triangles.

$$\begin{tikzcd}
\Sigma^{-1}U \arrow[r] \arrow[d, "\Sigma^{-1}\beta"'] & X \arrow[r, "f"] \arrow[d, "k"'] & V \arrow[r, "\alpha"] \arrow[d, "="'] \arrow[dl, dashed, "h"'] & U \arrow[d, "\beta"'] \\
\Sigma^{-1}W \arrow[r] & Z \arrow[r, "g"'] & V \arrow[r, "\beta \circ \alpha"'] & W
\end{tikzcd}$$

As $\mathcal{V}$ is cosuspended $\Sigma^{-1}W$ and Z are objects of $\mathcal{V}$ and since $f : X \to V$ is a $\mathcal{V}$-preenvelope, the morphism $k : X \to Z$ factors (not necessarily uniquely) via f, say $k = h \circ f$. Set $e = g \circ h : V \to V$ and note that $e \circ f = f$. We will show that e is an idempotent. First we prove the following claim:

Claim: If $l : V \to V'$, with $V' \in \mathcal{V}$, satisfies $l \circ f = 0$, then $l \circ e = 0$.

Indeed, since $l \circ f = 0$, l factors via α say $l = l' \circ \alpha$ and since $\beta : U \to W$ is a $\mathcal{V}$-preenvelope and $V' \in \mathcal{V}$, l' factors via β, say $l' = l'' \circ \beta$. So $l = l'' \circ (\beta \circ \alpha)$ which implies that $l \circ g = 0$. In particular, $l \circ e = l \circ g \circ h = 0$ and we have proven the claim.

Now since $e \circ f = e$, $(1-e) \circ f = 0$ and $(1-e) : V \to V$ where $V \in \mathcal{V}$, so applying the claim we get $(1-e) \circ e = 0$ or equivalently $e^2 = e$. Since idempotents in $\mathcal{T}$ split (as $\mathcal{T}$ has coproducts- see [43, see Proposition 1.6.8]) we can write $e = t \circ s$ where $s : V \to \hat{V}$, $t : \hat{V} \to V$ and $s \circ t = \mathrm{id}_{\hat{V}}$. Since $\mathcal{V}$ is closed under direct summands, $\hat{V} \in \mathcal{V}$. We set $\hat{V} = \lambda(X)$ and claim that $\varepsilon_X = s \circ f : X \to \lambda(X)$ is universal among morphisms from X to an object in $\mathcal{V}$. Indeed, if $f' : X \to V'$ where $V' \in \mathcal{V}$ then $f' = f'' \circ f$ as f is a $\mathcal{V}$-preenvelope. But then $f' = f'' \circ e \circ f = f'' \circ t \circ s \circ f = f'' \circ t \circ \varepsilon_X$. Now if there exists some $l : \lambda(X) \to V'$ such that $f' = l \circ \varepsilon_X = l \circ s \circ f$ then $(f'' \circ t - l) \circ s \circ f = 0$. Therefore, by the claim $(f'' \circ t - l) \circ s \circ e = 0$ and since $s \circ e = s$ we get $(f'' \circ t - l) \circ s = 0$. Finally precomposing with t gives $f'' \circ t - l = 0$ and the factorisation is unique, as required. $\square$

Corollary 5.2.7. *[33, Definition and Proposition 1.1] Let $\mathcal{V}$ be a suspended and precovering (respectively cosuspended and preenveloping) subcategory of $\mathcal{T}$. Then $(\mathcal{V}, \mathcal{V}^{\perp})$ (respectively $({}^{\perp}\mathcal{V}, \mathcal{V})$) forms a t-structure.*

Proof. See [33, Definition and Proposition 1.1] for the suspended case. The cosuspended case is dual. For completeness we sketch the proof of the cosuspended case.

We need to show that for every object $X \in \mathcal{T}$, there exists a distinguished triangle

$$X' \to X \to X'' \to \Sigma X'$$

such that $X' \in {}^{\perp}\mathcal{V}$ and $X'' \in \mathcal{V}$. As $\mathcal{V}$ is cosuspended, the inclusion $\mathcal{V} \hookrightarrow \mathcal{T}$ has a left adjoint by Proposition 5.2.6. Let us denote it by $\lambda : \mathcal{T} \to \mathcal{V}$, so $\mathcal{V}(\lambda(X), Y) \cong \mathcal{T}(X, Y)$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{V}$. Let X be an object of $\mathcal{T}$ and $\varepsilon_X : X \to \lambda(X)$ be the map corresponding to $\mathrm{id}_{\lambda(X)}$ under the above isomorphism. Complete ε_X to an exact triangle in $\mathcal{T}$,

$$Z \xrightarrow{\mu} X \xrightarrow{\varepsilon_X} \lambda(X) \xrightarrow{\gamma} \Sigma Z.$$

We want to show that $Z \in {}^{\perp}\mathcal{V}$. Suppose $h : Z \to V'$ is a morphism in $\mathcal{T}$, where $V' \in \mathcal{V}$. Consider the following morphism of triangles.

$$\begin{tikzcd}
\Sigma^{-1}\lambda(X) \arrow[r, "\Sigma^{-1}\gamma"] \arrow[d, "="'] & Z \arrow[r, "\delta"] \arrow[d, "h"'] & X \arrow[r, "\varepsilon_X"] \arrow[d, "l"'] & \lambda(X) \arrow[d, "="'] \arrow[dl, dashed, "s"'] \\
\Sigma^{-1}\lambda(X) \arrow[r, "h \circ \Sigma^{-1}\gamma"'] & V' \arrow[r] & Y \arrow[r, "k"'] & \lambda(X)
\end{tikzcd}$$

Since $\mathcal{V}$ is closed under extensions, $Y \in \mathcal{V}$ and therefore, as shown on the diagram, the morphism l factors uniquely through $\varepsilon_{\mathcal{X}}$. Notice that $k \circ s \circ \varepsilon_X = k \circ l = \varepsilon_X$. Therefore $k \circ s = \mathrm{id}_{\lambda(X)}$ by the universal property of the left adjoint λ. Therefore $h \circ \Sigma^{-1}\gamma = h \circ \Sigma^{-1}(\gamma \circ k \circ s) = 0$ and h factors through δ, say $h = h' \circ \delta$. But then since $V' \in \mathcal{V}$, $h' : X \to V'$ factors via ε_X, say $h' = h'' \circ \varepsilon_X$. Putting this together we get $h = h'' \circ \varepsilon_X \circ \delta = 0$ and $Z \in {}^{\perp}\mathcal{V}$, as required. $\square$

Next we recall some results from [37] which relate to triangulated definable subcategories. We start with some definitions.

Definition 5.2.8. [37, Corollary 12.6] An ideal $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ is said to be **exact** if the following three conditions hold:

(i) $\mathcal{J}$ is shift-closed,

(ii) $\mathcal{J}$ is cohomological,

(iii) $\mathcal{J}$ is idempotent i.e. for every $f \in \mathcal{J}$ there exist morphisms $g, h \in \mathcal{J}$ such that $f = h \circ g$.

Next we characterise the Serre subcategories of mod-$\mathcal{T}^c$ which correspond to exact ideals.

Definition 5.2.9. A Serre subcategory $\mathsf{C} \subseteq$ mod-$\mathcal{T}^c$ is said to be **perfect** if the right adjoint to the quotient functor Mod-$\mathcal{T}^c \to$ Mod-$\mathcal{T}^c/\varinjlim \mathsf{C}$ is an exact functor.

A Serre subcategory $\mathsf{C} \subseteq$ mod-$\mathcal{T}^c$ is said to be **shift-closed** if $G_f \in \mathsf{C}$ if and only if $G_{\Sigma f} \in \mathsf{C}$.

Theorem 5.2.10. *[37] There is a bijective correspondence between the following:*

(i) the exact ideals $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$*,*

(ii) the smashing subcategories $\mathcal{B} \subseteq \mathcal{T}$*,*

(iii) the triangulated definable subcategories $\mathcal{D} \subseteq \mathcal{T}$*,*

(iv) the shift-closed perfect Serre subcategories $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$*.*

Here the bijection between (ii) and (iii) is given by mapping a smashing subcategory $\mathcal{B}$ *to the definable subcategory* $\mathcal{D} = \mathcal{B}^{\perp}$*, and a definable subcategory* $\mathcal{D}$ *to the smashing subcategory* $\mathcal{B} = {}^{\perp}\mathcal{D}$*.*

Proof. (i) $\leftrightarrow$ **(ii)**: This is [37, Corollary 12.5]. The correspondence is given by

$$\mathcal{J} \mapsto \{X \in \mathcal{T} : \text{ every morphism } C \to X \text{ with } C \in \mathcal{T}^c \text{ factors via some } f \in \mathcal{J}\}$$

with inverse

$$\mathcal{B} \mapsto \{f \in \mathrm{morph}(\mathcal{T}^c) : f \text{ factors via some } X \in \mathcal{B}\}.$$

(i) $\leftrightarrow$ **(iv)**: This is [37, Proposition 8.8]. The correspondence is given by

$$\mathcal{J} \mapsto \{M \in \text{mod-}\mathcal{T}^c : M \cong \mathrm{im}\ H_f, \quad \text{for some } f \in \mathcal{J}\} \subseteq \text{mod-}\mathcal{T}^c$$

with inverse

$$\mathsf{C} \mapsto \{f \in \mathrm{morph}(\mathcal{T}^c) : \mathrm{im}\ H_f \in \mathsf{C}\}.$$

(ii) $\leftrightarrow$ **(iii)**: Suppose $\mathcal{B}$ is a smashing subcategory of $\mathcal{T}$. By [37, Lemma 12.4], $\mathcal{B}^{\perp}$ is a triangulated definable subcategory. Conversely, if $\mathcal{D}$ is a definable and triangulated subcategory of $\mathcal{T}$ then $({}^{\perp}\mathcal{D}, \mathcal{D})$ is a torsion pair by Corollary 5.2.7. Since $\mathcal{D}$ is closed under coproducts, ${}^{\perp}\mathcal{D}$ is a smashing subcategory by the well known alternative characterisation given in Proposition 5.2.4. The fact that these assignments give inverse bijections is a direct consequence of $({}^{\perp}\mathcal{D}, \mathcal{D})$ being a t-structure. $\square$

Remark 5.2.11. Let $\delta : \mathrm{Coh}(\mathcal{T}) \xrightarrow{\sim} (\text{mod-}\mathcal{T}^c)^{\mathrm{op}}$ be the duality defined in Proposition 5.1.2. By Lemma 5.1.4, the correspondence between (i) and (iv) is such that the Serre subcategory $\mathsf{C} = \delta\mathsf{S}$ where the $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ corresponds to $\mathcal{J}$ as in Theorem 2.5.11. In addition, [35, Theorem 4.2] tells us that the exact ideal associated as in (i) $\leftrightarrow$ (ii) to a smashing subcategory $\mathcal{B}$ is equal to the cohomological ideal associated as in Theorem 2.5.11 to the triangulated definable subcategory $\mathcal{D} = \mathcal{B}^{\perp}$. In this sense the correspondences in Theorem 5.2.10 are a restriction of the correspondences given in Krause's Fundamental Correspondence (Theorem 2.5.11) to triangulated definable subcategories.

The next result describes the relationship between a smashing subcategory $\mathcal{B}$ and the perfect Serre subcategory $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ associated by Theorem 5.2.10.

Proposition 5.2.12. *[35, Lemma 3.9] Let $\mathcal{B}$ be a smashing subcategory of $\mathcal{T}$ and let C be the corresponding perfect Serre subcategory from Theorem 5.2.10.*

Then $X \in \mathcal{B}$ if and only if $H_X \in \varinjlim \mathsf{C}$.

Now let us consider what happens when $\mathcal{D}$ is not just triangulated and definable but a definable tensor-ideal.

Proposition 5.2.13. *There is a bijective correspondence between the following.*

(i) The smashing tensor-ideals $\mathcal{B} \subseteq \mathcal{T}$.

(ii) The definable tensor-ideals $\mathcal{D} \subseteq \mathcal{T}$.

The bijection is given by $\mathcal{B} \mapsto \mathcal{B}^{\perp}$ and $\mathcal{D} \mapsto {}^{\perp}\mathcal{D}$. In particular, a smashing tensor-ideal $\mathcal{B}$ and its associated triangulated definable subcategory $\mathcal{D}$ fit into a torsion pair of the form $(\mathcal{B}, \mathcal{D})$.

Proof. Let $(\mathcal{B}, \mathcal{D})$ be a torsion pair with $\mathcal{B}$ a smashing subcategory and $\mathcal{D}$ definable and triangulated. $\mathcal{D}$ is $\mathcal{T}^c$-tensor-closed if and only if for every $Y \in \mathcal{B}$, $X \in \mathcal{D}$ and $C \in \mathcal{T}^c$, $(Y, C \otimes X) = 0$ if and only if for every $Y \in \mathcal{B}$, $X \in \mathcal{D}$ and $C \in \mathcal{T}^c$, $(C^{\vee} \otimes Y, X) = 0$ if and only if $\mathcal{B}$ is $\mathcal{T}^c$-tensor-closed. By Theorem 5.1.8, $\mathcal{D}$ is $\mathcal{T}$-tensor-closed (and consequently a definable tensor-ideal) if and only if it is $\mathcal{T}^c$-tensor-closed. It remains to show that $\mathcal{B}$ is $\mathcal{T}^c$-tensor-closed if and only if it is a smashing tensor-ideal.

If $\mathcal{B}$ is a smashing tensor-ideal it is $\mathcal{T}^c$-tensor-closed. Conversely, suppose that $\mathcal{B}$ is $\mathcal{T}^c$-tensor-closed and consider the family of coproduct preserving exact functors $\{X \otimes - : \mathcal{T} \to \mathcal{T} : X \in \mathcal{B}\}$. Applying [56, Lemma 3.8] with $\mathcal{M} = \mathcal{B}$ we get that $\mathcal{L} = \{X \in \mathcal{T} : Y \otimes X \in \mathcal{B},\ \forall Y \in \mathcal{B}\}$ is a localising subcategory of $\mathcal{T}$. But since $\mathcal{B}$ is $\mathcal{T}^c$-tensor-closed, $\mathcal{T}^c \subseteq \mathcal{L}$ meaning $\mathcal{L} = \mathcal{T}$. Therefore $\mathcal{B}$ is $\mathcal{T}$-tensor-closed and hence a smashing tensor-ideal, as required. $\square$

In summary, combining Theorem 5.2.10, with Theorem 5.1.8 and Proposition 5.2.13, we have the following triangulated version of Theorem 5.1.8.

Theorem 5.2.14. *Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category, $\mathcal{D}$ be a definable subcategory of $\mathcal{T}$, S be the corresponding Serre subcategory of $\mathrm{Coh}(\mathcal{T})$, $\mathsf{C} = \delta\mathsf{S}$ be the Serre subcategory of $\mathrm{mod}\text{-}\mathcal{T}^c$ given by applying δ to every functor in S and $\mathcal{J}$ be the corresponding cohomological ideal of morphisms in $\mathcal{T}^c$ (see 2.5.11). The following are equivalent:*

(i) $\mathcal{D}$ is a tensor-ideal, that is $\mathcal{T}$-tensor-closed and triangulated;

(ii) C is a perfect Serre tensor-ideal;

(iii) $\mathcal{J}$ is exact and $\mathcal{T}^c$-tensor-closed.

In addition, the above equivalent conditions hold if and only if $\mathcal{B} = {}^{\perp}\mathcal{D}$ is a smashing tensor-ideal of $\mathcal{T}$.

Let us denote the lattice of definable tensor-ideals of $\mathcal{T}$ by $(\mathbb{D}^{\otimes\Delta}(\mathcal{T}), \subseteq)$ and the lattice of smashing tensor-ideals of $\mathcal{T}$ by $(\mathbb{S}^{\otimes}(\mathcal{T}), \subseteq)$.

Corollary 5.2.15. *Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category. There is a lattice isomorphism $(\mathbb{S}^{\otimes}(\mathcal{T}), \subseteq) \cong (\mathbb{D}^{\otimes\Delta}(\mathcal{T}), \subseteq)^{\mathrm{op}}$.*

Proof. Note that if $\mathcal{D} \subseteq \mathcal{D}'$ and $X \in {}^{\perp}\mathcal{D}'$ then for all $Y \in \mathcal{D}$, $(X, Y) = 0$ so $X \in {}^{\perp}\mathcal{D}$. Hence ${}^{\perp}\mathcal{D}' \subseteq {}^{\perp}\mathcal{D}$. Therefore, the one-to-one correspondence between definable tensor-ideals and smashing tensor-ideals is inclusion-reversing. $\square$

In the remainder of this section we consider the role of localisation functors.

Definition 5.2.16. [38, Definition 2.4] A triangulated functor (L, α) with $L : \mathcal{T} \to \mathcal{T}$ and $\alpha : L \circ \Sigma \xrightarrow{\sim} \Sigma \circ L$, is a **localisation functor** if there exists a natural transformation $\eta : \mathrm{Id}_{\mathcal{T}} \to L$ such that $\Sigma\eta_X = \alpha_X \circ \eta_{\Sigma X}$, $L\eta : L \to L^2$ is invertible and $L\eta = \eta L$. **Colocalisation functors** are defined dually.

A localisation functor L is said to be a **smashing localisation** if L preserves coproducts.

We have the following characterisation of a smashing subcategory.

Proposition 5.2.17. *(e.g. [46, Proposition 4.4.3]) A full triangulated subcategory $\mathcal{B} \subseteq \mathcal{T}$ is smashing if and only if it is the kernel of a smashing localisation functor $L : \mathcal{T} \to \mathcal{T}$.*

If $\mathcal{B} \subseteq \mathcal{T}$ is a smashing subcategory, we can take L to be the composition $\mathcal{T} \xrightarrow{\lambda} \mathcal{B}^{\perp} \xhookrightarrow{i} \mathcal{T}$, where i is the inclusion functor and λ is left adjoint to i.

Proposition 5.2.18. *(e.g. [10, Theorem 2.6]) To every smashing localisation functor L there corresponds a colocalisation functor Γ such that, for every $X \in \mathcal{T}$ there exists a distinguished triangle, $\Gamma(X) \to X \to L(X) \to \Sigma\Gamma(X)$. In this case, $\mathcal{B} = \ker(L) = \mathrm{im}(\Gamma)$, $\mathcal{B}^{\perp} = \mathrm{im}(L) = \ker(\Gamma)$ and $\mathcal{B} = {}^{\perp}(\mathcal{B}^{\perp})$.*

So by Theorem 5.2.10, any triangulated definable subcategory, $\mathcal{D}$, can be written as $\mathcal{B}^{\perp} = \mathrm{im}(L) = \ker(\Gamma)$ for some smashing subcategory $\mathcal{B}$ with corresponding smashing localisation and colocalisation functors L and Γ respectively.

Next we prove a result from [31] which says that if the kernel of a localisation functor L is a tensor-ideal, then L is smashing if and only if $L \cong L(1) \otimes -$. First we prove the following two lemmas.

Lemma 5.2.19. *Let $(L : \mathcal{T} \to \mathcal{T}, \eta : \mathrm{Id}_{\mathcal{T}} \to L)$ be a localisation functor. Then for all $X,\ Y \in \mathcal{T}$, $(\eta_X, L(Y)) : (L(X), L(Y)) \xrightarrow{\sim} (X, L(Y))$ is an isomorphism.*

Proof. We claim that the map $\theta : (X, L(Y)) \to (L(X), L(Y))$ which takes a morphism $f : X \to L(Y)$ to $\eta^{-1}_{L(X)} \circ L(f) : L(X) \to L(L(Y)) \to L(Y)$, defines an inverse for $(\eta_X, L(Y))$. Indeed, since η is a natural transformation, for every $g : L(X) \to L(Y)$, we have $\eta_{L(Y)} \circ g = L(g) \circ \eta_{L(X)}$. But note that $\eta_{L(Y)} = (\eta L)(Y) = (L\eta)(Y)$ is invertible so

$$g = \eta^{-1}_{L(Y)} \circ L(g) \circ \eta_{L(X)} = \eta^{-1}_{L(Y)} \circ L(g) \circ L(\eta_X) = \eta^{-1}_{L(Y)} \circ L(g \circ \eta_X) = (\theta \circ (\eta_X, L(Y)))(g).$$

Similarly, for any $f : X \to L(Y)$ we have that $\eta_{L(Y)} \circ f = L(f) \circ \eta_X$, so

$$f = \eta_{L(Y)}^{-1} \circ L(f) \circ \eta_X = (\eta_X, L(Y))(\eta_{L(Y)}^{-1} \circ L(F)) = ((\eta_X, L(Y)) \circ \theta)(f),$$

as required. $\square$

Lemma 5.2.20. *[31, Lemma 3.1.6 (b)] Let $(L : \mathcal{T} \to \mathcal{T}, \eta : \mathrm{Id}_{\mathcal{T}} \to L)$ be a localisation functor such that for every $X,\ Y \in \mathcal{T}$, if $L(X) = 0$ then $L(X \otimes Y) = 0$ and suppose $C \in \mathcal{T}^c$. Then $\eta_{L(1)\otimes C} : L(1) \otimes C \to L(L(1) \otimes C)$ is an isomorphism.*

Proof. First we show that if $L(Z) = 0$ then $(Z, L(1) \otimes C) = 0$ for any $Z \in \mathcal{T}$. Suppose that $L(Z) = 0$. Then $(Z, L(1) \otimes C) \cong (C^\vee \otimes Z, L(1)) \cong (L(C^\vee \otimes Z), L(1))$ by Lemma 5.2.19. But $L(Z) = 0$ so $L(C^\vee \otimes Z) = 0$ by the assumption on L. Therefore $(Z, L(1) \otimes C) = 0$, as required.

Let

$$\Gamma(L(1) \otimes C) \xrightarrow{\gamma} L(1) \otimes C \xrightarrow{\eta_{L(1)\otimes C}} L(L(1) \otimes C) \to \Sigma\Gamma(L(1) \otimes C)$$

be the exact triangle as in Proposition 5.2.18. As $L(\Gamma(L(1) \otimes C)) = 0$, $(\Gamma(L(1) \otimes C), L(1) \otimes C) = 0$ meaning $\gamma : \Gamma(L(1) \otimes C) \to L(1) \otimes C$ is the zero morphism. Consequently, there exists some $\mu : L(L(1) \otimes C) \to L(1) \otimes C$ such that $\mathrm{id}_{L(1)\otimes C} = \mu \circ \eta_{L(1)\otimes C}$. Applying L to this equation we get $\mathrm{id}_{L(L(1)\otimes C)} = L(\mu) \circ L(\eta_{L(1)\otimes C}) = L(\mu) \circ \eta_{L(L(1)\otimes C)}$. Since η is a natural transformation, $L(\mu) \circ \eta_{L(L(1)\otimes C)} = \eta_{L(1)\otimes C} \circ \mu$. So μ is inverse to $\eta_{L(1)\otimes C}$ as required. $\square$

Proposition 5.2.21. *[31, Definition 3.3.2] Let $L : \mathcal{T} \to \mathcal{T}$ be a localisation functor such that for every $X,\ Y \in \mathcal{T}$, if $L(X) = 0$ then $L(X \otimes Y) = 0$. Then the following are equivalent:*

- *There exists a natural isomorphism $\beta : L(1) \otimes - \to L$;*
- *L preserves coproducts.*

Proof. Suppose there exists a natural isomorphism $\beta : L(1) \otimes - \to L$. Then

$$L(\coprod_{i \in I} X_i) \cong L(1) \otimes (\coprod_{i \in I} X_i) \cong \coprod_{i \in I} (L(1) \otimes X_i) \cong \coprod_{i \in I} L(X_i).$$

Conversely, suppose L preserves coproducts. For $X \in \mathcal{T}$, consider the distinguished triangle $L(\Gamma(1) \otimes X) \xrightarrow{L(\gamma_X)} L(X) \xrightarrow{L(\lambda_X)} L(L(1) \otimes X) \to \Sigma L(\Gamma(1) \otimes X)$. By [10, Theorem 2.6] $L(\Gamma(1)) = 0$ so by assumption $L(\Gamma(1) \otimes X) = 0$. Hence $L(\lambda_X) : L(X) \xrightarrow{\sim} L(L(1) \otimes X)$ is an isomorphism. Define the map $\beta_X : L(1) \otimes X \to L(X)$ by

$$L(1) \otimes X \xrightarrow{\eta_{L(1) \otimes X}} L(L(1) \otimes X) \xrightarrow{L(\lambda_X)^{-1}} L(X).$$

Let $\mathcal{A} \subseteq \mathcal{T}$ denote the full subcategory of $\mathcal{T}$ given by those $X \in \mathcal{T}$ such that β_X is an isomorphism. We show that $\mathcal{T}^c \subseteq \mathcal{A}$ and that $\mathcal{A}$ is a localising subcategory of $\mathcal{T}$, therefore $\mathcal{A} = \mathcal{T}$.

If $\beta_X : L(1) \otimes X \to L(X)$ is an isomorphism then so is $\eta_{L(1) \otimes X} : L(1) \otimes X \to L(L(1) \otimes X)$. But then $\Sigma(\eta_{L(1) \otimes X})$ is also an isomorphism, so since $\Sigma(\eta_{L(1) \otimes X}) \simeq \eta_{\Sigma(L(1) \otimes X)} \simeq \eta_{L(1) \otimes \Sigma X}$, we have $\beta_{\Sigma X}$ is an isomorphism and therefore $\Sigma X \in \mathcal{A}$.

Suppose $X \to Y \to Z \to \Sigma X$ is a distinguished triangle in $\mathcal{T}$ and $X,\ Z \in \mathcal{A}$. Then we have the following commutative diagram where the rows are distinguished triangles.

$$\begin{tikzcd}
L(1) \otimes X \arrow[r] \arrow[d, "\alpha_X"'] & L(1) \otimes Y \arrow[r] \arrow[d, "\alpha_Y"'] & L(1) \otimes Z \arrow[r] \arrow[d, "\alpha_Z"'] & L(1) \otimes \Sigma X \arrow[d, "\alpha_{\Sigma X}"'] \\
L(X) \arrow[r] & L(Y) \arrow[r] & L(Z) \arrow[r] & L(\Sigma X)
\end{tikzcd}$$

Since β_X and β_Z are isomorphisms, so is β_Y. Therefore $Y \in \mathcal{A}$ and $\mathcal{A}$ is triangulated.

Let us check that $\mathcal{A}$ is closed under arbitrary coproducts. Suppose $\{X_i\}_{i \in I} \subseteq \mathcal{A}$ and consider

$$\beta_{\coprod_i X_i} = L(\lambda_{\coprod_i X_i})^{-1} \circ \eta_{L(1) \otimes \coprod_i X_i} : L(1) \otimes \coprod_i X_i \to L(\coprod_i X_i).$$

We need to show that $\eta_{L(1)\otimes\coprod_i X_i}$ is an isomorphism. Since for each $j \in I$, $X_j \in \mathcal{A}$, we have that for each $j \in I$, $\eta_{L(1)\otimes X_j}$ is an isomorphism. Therefore we have maps, $L(L(1)\otimes X_j) \xrightarrow{\eta^{-1}_{L(1)\otimes X_j}} L(1)\otimes X_j \xrightarrow{i_{L(1)\otimes X_j}} \coprod_i (L(1)\otimes X_i)$ and by the universal property of coproducts there exists a unique map $\mu : \coprod_i L(L(1)\otimes X_i) \to \coprod_i (L(1)\otimes X_i)$ making the following diagram commute.

$$
\begin{tikzcd}
L(L(1)\otimes X_j) \arrow[d, "\eta^{-1}_{L(1)\otimes X_j}"'] \arrow[ddrr, "i_{L(L(1)\otimes X_j)}"] & & \\
L(1)\otimes X_j \arrow[d, "i_{L(1)\otimes X_j}"'] & & \\
\coprod_i (L(1)\otimes X_i) & & \coprod_i L(L(1)\otimes X_i) \arrow[ll, "\mu"']
\end{tikzcd}
$$

Now, since $L(1)\otimes -$ and L preserve coproducts we have isomorphisms $\gamma : \coprod_i L(L(1)\otimes X_i) \to L(L(1)\otimes \coprod_i X_i)$ and $\xi : \coprod_i (L(1)\otimes X_i) \to L(1)\otimes \coprod_i X_i$ such that

$$\gamma \circ i_{L(L(1)\otimes X_j)} = L(L(1)\otimes i_{X_j})$$

and

$$\xi \circ i_{L(1)\otimes X_j} = L(1)\otimes i_{X_j}.$$

We will show that the composition

$$L(L(1)\otimes \coprod_i X_i) \xrightarrow{\beta^{-1}} \coprod_i L(L(1)\otimes X_i) \xrightarrow{\mu} \coprod_i (L(1)\otimes X_i) \xrightarrow{\delta} (L(1)\otimes \coprod_i X_i)$$

is inverse to $\eta_{L(1)\otimes\coprod_i X_i}$. We have the following commutative diagram.

$$\begin{tikzcd}
\coprod_i (L(1)\otimes X_i) \arrow[dd, "\xi"'] & & & \coprod_i L(L(1)\otimes X_i) \arrow[lll, "\mu"'] \arrow[dd, "\gamma"] \\
& L(1)\otimes X_j \arrow[ul, "i_{L(1)\otimes X_j}"'] \arrow[dl, "L(1)\otimes i_{X_j}"] \arrow[r, bend right, "\eta_{L(1)\otimes X_j}"'] & L(L(1)\otimes X_j) \arrow[l, bend right, "\eta^{-1}_{L(1)\otimes X_j}"'] \arrow[ur, "i_{L(L(1)\otimes X_j)}"] \arrow[dr, "L(L(1)\otimes i_{X_j})"'] & \\
(L(1)\otimes \coprod_i X_i) \arrow[rrr, "\eta_{L(1)\otimes \coprod_i X_i}"] & & & L(L(1)\otimes \coprod_i X_i)
\end{tikzcd}$$

It can then easily be checked, through some diagram chasing, that for all $j \in I$,

$$\gamma^{-1} \circ \eta_{L(1)\otimes \coprod_i X_i} \circ \xi \circ \mu \circ i_{L(L(1)\otimes X_j)} = i_{L(L(1)\otimes X_j)}$$

and

$$\mu \circ \gamma^{-1} \circ \eta_{L(1)\otimes \coprod_i X_i} \circ \xi \circ i_{L(1)\otimes X_j} = i_{L(1)\otimes X_j}.$$

Therefore $\gamma^{-1} \circ \eta_{L(1)\otimes \coprod_i X_i} \circ \xi \circ \mu = \mathrm{id}_{\coprod_i L(L(1)\otimes X_i)}$ and $\mu \circ \gamma^{-1} \circ \eta_{L(1)\otimes \coprod_i X_i} \circ \xi = \mathrm{id}_{\coprod_i (L(1)\otimes X_i)}$. Consequently, $\xi\circ\mu\circ\gamma^{-1}$ is the inverse of $\eta_{L(1)\otimes \coprod_i X_i}$. Hence $\coprod_i X_i \in \mathcal{A}$ as required.

Finally, note that by Lemma 5.2.20, $\eta_{L(1)\otimes C} : L(1)\otimes C \to L(L(1)\otimes C)$ is an isomorphism for all $C \in \mathcal{T}^c$, so $\mathcal{T}^c \subseteq \mathcal{A}$. $\square$

In summary we have seen that, if $\mathcal{D}$ is a definable tensor-ideal, then $\mathcal{D}$ is the coaisle of a torsion pair $(\mathcal{B}, \mathcal{D})$ where $\mathcal{B}$ is a smashing tensor-ideal (Proposition 5.2.13). Furthermore, the inclusion functor $\mathcal{D} \overset{i}{\hookrightarrow} \mathcal{T}$ has a left adjoint, $\lambda : \mathcal{T} \to \mathcal{D}$ and $L = i \circ \lambda$ defines a smashing localisation functor. Therefore $L(1) = \lambda(1)$, $\mathcal{D} = \mathrm{im}(\lambda(1)\otimes -)$ and $\mathcal{B} = \ker(\lambda(1)\otimes -)$. Set $\mathsf{C} = \delta\mathsf{S}$ where $\mathsf{S} \subseteq \mathrm{Coh}(\mathcal{T})$ is the Serre subcategory associated to $\mathcal{D}$ as in Theorem 2.5.11. Then by Proposition 5.2.12, $X \in \mathcal{B}$ if and only if $H_X \in \varinjlim \mathsf{C}$. Recall that $\overline{H} : \mathcal{T} \to \text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ is given by the composition $\mathcal{T} \xrightarrow{H} \text{Mod-}\mathcal{T}^c \xrightarrow{q} \text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$. Let $\overline{H}_1$ be the image of the tensor unit in $\mathcal{T}$ under $\overline{H}$ and suppose that the injective hull of $\overline{H}_1$ is given by $\overline{H}_E$ for a pure-injective $E \in \mathcal{T}$. By [12, Theorem 3.5] (see the end of Section 5.1) $\varinjlim \mathsf{C} = \ker(H_E \otimes -)$ and therefore $\mathcal{B} = \ker(E\otimes -)$. The following proposition shows that E is isomorphic to the pure-injective hull of $\lambda(1)$.

Proposition 5.2.22. *Suppose $\mathcal{D}$ is $\mathcal{T}$-tensor-closed and definable and $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ is the corresponding Serre subcategory. If the inclusion $\mathcal{D} \hookrightarrow \mathcal{T}$ has a left adjoint, $\lambda : \mathcal{T} \to \mathcal{D}$, then the (unique up to unique isomorphism) pure-injective $E \in \mathcal{T}$ constructed in [12, Construction 3.1] is isomorphic to the pure-injective hull of $\lambda(1)$.*

Proof. Let $\eta_\lambda : \lambda(1) \to E_\lambda$ denote the pure-injective hull of $\lambda(1)$ in $\mathcal{T}$ and note that $\lambda(1) \in \mathcal{D}$ implies $E_\lambda \in \mathcal{D}$. Let $\overline{H} : \mathcal{T} \to \text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ denote H composed with the quotient map $q : \text{Mod-}\mathcal{T}^c \to \text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ and let $\overline{H}_\eta : \overline{H}_1 \to \overline{H}_E$ be the injective hull of $\overline{H}_1$ in the quotient category $\text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ as defined in Construction 3.1 of [12] (see Section 5.1). Let $\varepsilon_1 : 1 \to \lambda(1)$ be the morphism in $\mathcal{T}$ corresponding to $\mathrm{id}_{\lambda(1)}$ under the adjunction isomorphism

$$\mathcal{D}(\lambda(1), \lambda(1)) \cong \mathcal{T}(1, \lambda(1)).$$

Since $\overline{H}_\eta : \overline{H}_1 \to \overline{H}_E$ is a monomorphism in $\text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ and $\overline{H}_{E_\lambda}$, the image of E_λ under $\overline{H}$, is injective, the map $\overline{H}_{\eta_\lambda} \circ \overline{H}_{\varepsilon_1}$ factors through $\overline{H}_\eta$.

$$\begin{tikzcd}
\overline{H}_1 \arrow[r, "\overline{H}_\eta"] \arrow[d, "\overline{H}_{\eta_\lambda} \circ \overline{H}_{\varepsilon_1}"'] & \overline{H}_E \arrow[dl, "\overline{H}_k"] \\
\overline{H}_{E_\lambda} &
\end{tikzcd}$$

Similarly, $\overline{H}_{\eta_\lambda}$ is a monomorphism in $\text{Mod-}\mathcal{T}^c/\varinjlim \mathsf{C}$ and $\overline{H}_E$ is injective so any map $\overline{H}_{\lambda(1)} \to \overline{H}_E$ factors via $\overline{H}_{\eta_\lambda}$. By Proposition 5.1.19, we have that $E \in \mathcal{D}$, so we have a morphism $\xi : \lambda(1) \to E$ corresponding to η under the adjunction isomorphism $\mathcal{T}(1, E) \cong \mathcal{D}(\lambda(1), E)$. By naturality of the adjunction we have $\eta = \xi \circ \varepsilon_1$. By the above observation we have a morphisms $\overline{k'} : \overline{E_\lambda} \to \overline{E}$ such that the following diagram commutes.

$$\begin{tikzcd}
\overline{H}_{\lambda(1)} \arrow[r, "\overline{H}_{\eta_\lambda}"] \arrow[d, "\overline{H}_{\xi}"'] & \overline{H}_{E_\lambda} \arrow[dl, "\overline{H}_{k'}"] \\
\overline{H}_{E} &
\end{tikzcd}$$

By [12, Corollary 2.18(c)] $\overline{H}_k$ and $\overline{H}_{k'}$ correspond to maps k and k' in $\mathcal{T}$. Now since,

$$\overline{H}_{k'} \circ \overline{H}_k \circ \overline{H}_\eta = \overline{H}_{k'} \circ \overline{H}_{\eta_\lambda} \circ \overline{H}_{\varepsilon_1} = \overline{H}_\xi \circ \overline{H}_{\varepsilon_1} = \overline{H}_\eta,$$

and $\overline{H}_\eta$ is an injective hull, we have that $\overline{H}_{k'} \circ \overline{H}_k$ is an automorphism. Applying [12, Corollary 2.18(c)] again, we get that $k' \circ k$ an automorphism. In a similar vein, $\overline{H}_{k' \circ \eta_\lambda - \xi} = \overline{H}_{k'} \circ \overline{H}_{\eta_\lambda} - \overline{H}_\xi = 0$ and since the target of this map, $\overline{H}_E$ is injective, $k' \circ \eta_\lambda = \xi$. By an identical argument, using that $\overline{H}_{E_\lambda}$ is injective, $k \circ \eta = \eta_\lambda \circ \varepsilon_1$. Therefore,

$$k \circ k' \circ \eta_\lambda \circ \varepsilon_1 = k \circ \xi \circ \varepsilon_1 = k \circ \eta = \eta_\lambda \circ \varepsilon_1.$$

Consider the following commutative diagram in **Ab** given by the naturality of the adjunction between the inclusion $\mathcal{D} \hookrightarrow \mathcal{T}$ and λ.

$$\begin{tikzcd}
\mathcal{D}(\lambda(1), \lambda(1)) \arrow[r, "\cong"] \arrow[d, "(\lambda(1), f)"'] & \mathcal{T}(1, \lambda(1)) \arrow[d, "(1, f)"] \\
\mathcal{D}(\lambda(1), Z) \arrow[r, "\cong"] & \mathcal{T}(1, Z)
\end{tikzcd}$$

Given a morphism $f : \lambda(1) \to Z$, $f \circ \varepsilon_1$ is the image of the identity on $\lambda(1)$ under the top horizontal isomorphism followed by $(1, f)$. But then $f \circ \varepsilon_1$ is isomorphic under the bottom horizontal isomorphism to f. So, if $f \circ \varepsilon_1 = 0$ then $f = 0$. Hence ε_1 is a monomorphism and since $k \circ k' \circ \eta_\lambda \circ \varepsilon_1 = \eta_\lambda \circ \varepsilon_1$, we have $k \circ k' \circ \eta_\lambda = \eta_\lambda$. As η_λ is a pure-injective hull, $k \circ k'$ is an automorphism. Suppose α is the two-sided inverse of $k' \circ k$ and β is the two sided inverse of $k \circ k'$. Then,

$$\alpha \circ k' = \alpha \circ k' \circ k \circ k' \circ \beta = k' \circ \beta,$$

is inverse to k on both sides. Hence k is an isomorphism and $E \cong E_\lambda$ as required.
$\square$

Chapter 6

Topologies associated to tensor triangulated categories

In this chapter we explore various topologies that can be associated to a rigidly-compactly generated tensor triangulated category $\mathcal{T}$. Fix a rigidly-compactly generated tensor triangulated category $\mathcal{T}$.

6.1 The Ziegler spectrum

In this section we define five new Ziegler-type topologies.

6.1.1 Shift-closed Ziegler topology

In this section we will define the positive shift-closed Ziegler topology, negative shift-closed Ziegler topology and shift-closed Ziegler topology on $\mathrm{pinj}_{\mathcal{T}}$ and show that the frame of open subsets of the shift-closed Ziegler topology is isomorphic to the frame of open subsets of a quotient topology of the Ziegler spectrum.

Lemma 6.1.1. *Let $\mathcal{D}$ be a definable subcategory of $\mathcal{T}$ with associated cohomological ideal $\mathcal{J}$ and $i \in \mathbb{Z}$. Then $\Sigma^i\mathcal{D} := \{\Sigma^i X : X \in \mathcal{D}\}$ is also definable in $\mathcal{T}$ with associated cohomological ideal $\Sigma^i\mathcal{J}$.*

Proof. Let $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ denote the cohomological ideal corresponding to $\mathcal{D}$ and set $\Sigma^i\mathcal{J} := \{\Sigma^i f : f \in \mathcal{J}\}$. The full subcategory, $\mathcal{X} \subseteq \mathcal{T}$, consisting of the $X \in \mathcal{T}$ satisfying $(f, X) = 0$ for all $f \in \Sigma^i\mathcal{J}$ is a definable subcategory. We show that $\mathcal{X} = \Sigma^i\mathcal{D}$. Indeed, $X \in \mathcal{D}$ if and only if $(f, X) = 0$ for all $f \in \mathcal{J}$ if and only if $(\Sigma^i f, \Sigma^i X) = 0$ for all $f \in \mathcal{J}$ if and only if $\Sigma^i X \in \mathcal{X}$.

So $\Sigma^i\mathcal{D}$ is a definable subcategory with corresponding cohomological ideal $\langle \Sigma^i\mathcal{J} \rangle^{\mathrm{cohom}}$. It remains to show that $\Sigma^i\mathcal{J} = \langle \Sigma^i\mathcal{J} \rangle^{\mathrm{cohom}}$. Indeed, $f \in \langle \Sigma^i\mathcal{J} \rangle^{\mathrm{cohom}}$ if and only if for every $X \in \mathcal{D}$, $(f, \Sigma^i X) = 0$ if and only if for every $X \in \mathcal{D}$, $(\Sigma^{-i} f, X) = 0$ if and only if $\Sigma^{-i} f \in \mathcal{J}$. So $\langle \Sigma^i\mathcal{J} \rangle^{\mathrm{cohom}} = \Sigma^i\mathcal{J}$ as required. $\square$

Definition 6.1.2. A definable subcategory $\mathcal{D}$ of $\mathcal{T}$ is said to be **positive shift-closed** (respectively **negative shift-closed**) if $X \in \mathcal{D}$ implies $\Sigma X \in \mathcal{D}$ (respectively $X \in \mathcal{D}$ implies $\Sigma^{-1} X \in \mathcal{D}$). A definable subcategory $\mathcal{D}$ of $\mathcal{T}$ is said to be **shift-closed** if it is both positive and negative shift-closed.

We will say that a cohomological ideal $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ is **positive shift-closed** (respectively **negative shift-closed**) if $f \in \mathcal{J}$ implies $\Sigma f \in \mathcal{J}$ (respectively $f \in \mathcal{J}$ implies $\Sigma^{-1} f \in \mathcal{J}$). We will say that a cohomological ideal $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ is **shift-closed** if it is both positive and negative shift-closed.

We will say that a Serre subcategory S of $\mathrm{Coh}(\mathcal{T})$ is **positive shift-closed** (respectively **negative shift-closed**) if $F_f \in \mathsf{S}$ implies $F_{\Sigma f} \in \mathsf{S}$ (respectively $F_f \in \mathsf{S}$ implies $F_{\Sigma^{-1} f} \in \mathsf{S}$). We will say that a Serre subcategory S of $\mathrm{Coh}(\mathcal{T})$ is **shift-closed** if it is both positive and negative shift-closed.

Corollary 6.1.3. *A definable subcategory, $\mathcal{D}$ of $\mathcal{T}$, is positive (respectively negative) shift-closed if and only if the corresponding cohomological ideal $\mathcal{J}$ of* $\mathrm{morph}(\mathcal{T}^c)$*, is negative (respectively positive) shift-closed if and only if the corresponding Serre subcategory* S *of* $\mathrm{Coh}(\mathcal{T})$ *is negative (respectively positive) shift-closed.*

Proof. $\mathcal{D}$ is positive shift-closed if and only if $\Sigma\mathcal{D} \subseteq \mathcal{D}$ if and only if $\mathcal{J}_{\mathcal{D}} \subseteq \mathcal{J}_{\Sigma\mathcal{D}}$. But by Lemma 6.1.1 $\mathcal{J}_{\Sigma\mathcal{D}} = \Sigma\mathcal{J}_{\mathcal{D}}$ so $\mathcal{D}$ is positive shift-closed if and only if $\mathcal{J}_{\mathcal{D}} \subseteq \Sigma\mathcal{J}_{\mathcal{D}}$ if and only if $f \in \mathcal{J}_{\mathcal{D}}$ implies $f \in \Sigma\mathcal{J}_{\mathcal{D}}$ (equivalently $\Sigma^{-1} f \in \mathcal{J}_{\mathcal{D}}$) if and only if $\mathcal{J}_{\mathcal{D}}$ is negative shift-closed. Similarly, $\mathcal{D}$ is negative shift-closed if and only if $\Sigma^{-1}\mathcal{D} \subseteq \mathcal{D}$ if and only if $\mathscr{J}_{\mathcal{D}} \subseteq \mathscr{J}_{\Sigma^{-1}\mathcal{D}} = \Sigma^{-1}\mathscr{J}_{\mathcal{D}}$ if and only if $\mathscr{J}_{\mathcal{D}}$ is positive shift-closed.

For the analogous statement regarding Serre subcategories consider Lemma 5.1.4. $\square$

Definition 6.1.4. Say that a subset $\mathscr{C} \subseteq \mathrm{pinj}_{\mathcal{T}}$ is **closed** with respect to the **positive shift-closed** (respectively **negative shift-closed**) (respectively **shift-closed**) **Ziegler topology** if it is of the form $\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ for some positive shift-closed (respectively negative shift-closed) (respectively shift-closed) definable subcategory $\mathcal{D}$.

Proposition 6.1.5. *The closed subsets in Definition 6.1.4 define a topology on* $\mathrm{pinj}_{\mathcal{T}}$ *in each of the three cases which we call the positive shift-closed Ziegler topology (respectively negative shift-closed Ziegler topology) (respectively shift-closed Ziegler topology) and denote by* $\mathrm{Zg}_{\mathcal{T}}^{\Sigma^+}$ *(respectively* $\mathrm{Zg}_{\mathcal{T}}^{\Sigma^-}$ *) (respectively* $\mathrm{Zg}_{\mathcal{T}}^{\Sigma}$*).*

Proof. We provide the proof in the positive shift-closed case. The proof in the negative shift-closed case is the same but with 'positive' and 'negative' interchanged. The proof in the shift-closed case is achieved by removing 'positive' and 'negative' from the proof below.

Suppose $\mathcal{D}$ and $\mathcal{D}'$ are positive shift-closed definable subcategories of $\mathcal{T}$. Then the definable subcategory generated by their union, $\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}}$, is positive shift-closed. Indeed, it corresponds to the Serre subcategory $\mathsf{S}_{\mathcal{D}} \cap \mathsf{S}_{\mathcal{D}'}$, where S_* is the Serre subcategory corresponding to $*$ for $* \in \{\mathcal{D}, \mathcal{D}'\}$. By Corollary 6.1.3, $\mathsf{S}_{\mathcal{D}}$ and $\mathsf{S}_{\mathcal{D}'}$ are negative shift-closed, so their intersection is also negative shift-closed. Applying Corollary 6.1.3 again gives the required result.

If we have a family $\{\mathcal{D}_i : i \in I\}$ of positive shift-closed definable subsets of $\mathcal{T}$, we know that the intersection $\bigcap_{i\in I} \mathcal{D}_i$ is definable. We must show that it is also positive shift-closed. We have $X \in \bigcap_{i\in I} \mathcal{D}_i$ if and only if $X \in \mathcal{D}_i$ for all $i \in I$, which implies that $\Sigma X \in \mathcal{D}_i$ for all $i \in I$ (since the $\mathcal{D}_i$s are positive shift-closed). But, $\Sigma X \in \mathcal{D}_i$ for all $i \in I$ if and only if $\Sigma X \in \bigcap_{i\in I} \mathcal{D}_i$, so $\bigcap_{i\in I} \mathcal{D}_i$ is positive shift-closed.

It remains to note the relationship between definable subcategories and their pure-injectives (see [36]). $\square$

Now let us consider a quotient topology of the Ziegler spectrum which we will show is equivalent (up to topologically indistinguishable objects) to the shift-closed Ziegler topology defined above.

Notation 6.1.6. Given a topological space X and an equivalence relation, $\sim$, on the set X, let $q : X \to X/_\sim$ denote the quotient map. We define the quotient topology on $X/_\sim$ to have open sets given by the subsets with open inverse image under q.

Remark 6.1.7. Any power of the shift functor applied to an indecomposable pure-injective gives an indecomposable pure-injective. This follows as Σ is an autoequivalence on $\mathcal{T}$ and the properties that an object of a compactly generated triangulated category are 'indecomposable' and 'pure-injective' are defined in terms of the category structure.

Definition 6.1.8. Define an equivalence relation on $\mathrm{pinj}_{\mathcal{T}}$ by $P \sim_\Sigma Q$ if and only if there exists $i \in \mathbb{Z}$ such that $P = \Sigma^i Q$. Let $q : \mathrm{pinj}_{\mathcal{T}} \to \mathrm{pinj}_{\mathcal{T}}/_{\sim_\Sigma}$ denote the quotient map. We denote the quotient of the Ziegler spectrum under $\sim_\Sigma$ by $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\Sigma}$.

Lemma 6.1.9. *A definable subcategory $\mathcal{D}$ of $\mathcal{T}$ satisfies $\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}} = q^{-1}(\mathscr{C})$ for some subset $\mathscr{C} \subseteq \mathrm{pinj}_{\mathcal{T}}/_{\sim_\Sigma}$ if and only if $\mathcal{D}$ is shift-closed.*

Proof. If $\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}} = q^{-1}(\mathscr{C})$ then $P \in \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ if and only if $[P]_{\sim_\Sigma} \in \mathscr{C}$ if and only if $\Sigma^i P \in \mathcal{D}$ for all $i \in \mathbb{Z}$. So the indecomposable pure-injectives in $\mathcal{D}$ are closed under shift. The corresponding cohomological ideal $\mathcal{J}$ is given by

$$\{f \in \mathrm{morph}(\mathcal{T}^c) : (f, P) = 0,\ \forall P \in \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}\}.$$

Therefore if $f \in \mathcal{J}$, then for any $i \in \mathbb{Z}$, and $P \in \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$, $(\Sigma^i f, P) \cong (f, \Sigma^{-i} P) = 0$ as $\Sigma^{-i} P \in \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$. Hence $f \in \mathcal{J}$ implies $\Sigma^i f \in \mathcal{J}$ for all $i \in \mathbb{Z}$ meaning $\mathcal{J}$ is shift-closed and by Corollary 6.1.3, $\mathcal{D}$ is shift-closed as required.

Conversely, suppose $\mathcal{D}$ is shift-closed and set $\mathscr{C} = q(\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}})$. Then $\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}} \subseteq q^{-1}(\mathscr{C})$ and if $P \in q^{-1}(\mathscr{C})$ then $P \sim_\Sigma Q$ for some $Q \in \mathcal{D}$. But then $P = \Sigma^n Q \in \mathcal{D}$ as $\mathcal{D}$ is shift-closed. Therefore $\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}} = q^{-1}(\mathscr{C})$ as required. $\square$

Given a topological space X, recall that we denote the frame of open subsets of X by $\mathbb{O}(X)$.

Proposition 6.1.10. *The quotient map q induces a frame isomorphism between $\mathbb{O}(\mathrm{Zg}^{\Sigma}_{\mathcal{T}})$ and $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}/_{\sim_\Sigma})$.*

Proof. Suppose $\mathscr{O} \subseteq \mathrm{pinj}_{\mathcal{T}}$ is an open subset of $\mathrm{Zg}^{\Sigma}_{\mathcal{T}}$. Then the closed complement $\mathscr{C} := \mathrm{pinj}_{\mathcal{T}} \backslash \mathscr{O} = \mathrm{pinj}_{\mathcal{T}} \cap \mathcal{D}$ for some shift-closed definable subcategory $\mathcal{D}$. As $\mathcal{D}$ is shift-closed, $[P]_{\sim_\Sigma} \notin q(\mathscr{C})$ if and only if for all $i \in \mathbb{Z}$, $\Sigma^i P \notin \mathcal{D}$ if and only if there exists $i \in \mathbb{Z}$, $\Sigma^i P \notin \mathcal{D}$ if and only if $[P]_{\sim_\Sigma} \in q(\mathscr{O})$. Therefore $q(\mathscr{O}) = (\mathrm{pinj}_{\mathcal{T}}/_{\sim_\Sigma}) \backslash q(\mathscr{C})$ is open.

Conversely, suppose $\mathscr{O}'$ is an open subset of $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\Sigma}$. Then by definition of the quotient topology, $q^{-1}(\mathscr{O}')$ is open in $\mathrm{pinj}_{\mathcal{T}}$ with respect to the Ziegler topology $\mathrm{Zg}_{\mathcal{T}}$. It is easy to see that the definable subcategory $\mathcal{D} \subseteq \mathcal{T}$ such that $\mathrm{pinj}_{\mathcal{T}} \backslash q^{-1}(\mathscr{O}') = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ coincides with the definable subcategory $\mathcal{D}'$ such that the closed complement $\mathcal{C}'$ in $\mathrm{pinj}_{\mathcal{T}}/_{\sim_\Sigma}$ of $\mathscr{O}'$ satisfies $q^{-1}(\mathcal{C}') = \mathcal{D}' \cap \mathrm{pinj}_{\mathcal{T}}$. Indeed, $P \in \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ if and only if $P \notin q^{-1}(\mathscr{O}')$ if and only if $q(P) \in \mathcal{C}'$ if and only if $P \in \mathcal{D}' \cap \mathrm{pinj}_{\mathcal{T}}$. By Lemma 6.1.9, $\mathcal{D}' = \mathcal{D}$ is shift-closed and therefore $q^{-1}(\mathscr{O}')$ is also open with respect to $\mathrm{Zg}^{\Sigma}_{\mathcal{T}}$.

It is straightforward to check that $\mathscr{O} \mapsto q(\mathscr{O})$ and $\mathscr{O}' \mapsto q^{-1}(\mathscr{O}')$ defines an order preserving bijection (and hence an isomorphism of frames) between the open subsets of $\mathrm{Zg}^{\Sigma}_{\mathcal{T}}$ and $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\Sigma}$ $\square$

Corollary 6.1.11. *P and Q are topologically indistinguishable in $\mathrm{Zg}^{\Sigma}_{\mathcal{T}}$ if and only if $[P]_{\sim_\Sigma}$ and $[Q]_{\sim_\Sigma}$ are topologically indistinguishable in $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\Sigma}$.*

6.1.2 $\mathcal{T}$-tensor-closed Ziegler topology

In this section we define a different Ziegler-type topology.

Definition 6.1.12. We say that a subset $\mathscr{C} \subseteq \mathrm{pinj}_{\mathcal{T}}$ is **closed** with respect to the **$\mathcal{T}$-tensor-closed Ziegler topology** if there exists a $\mathcal{T}$-tensor-closed definable subcategory $\mathcal{D}$ of $\mathcal{T}$ such that $\mathscr{C} = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$.

Proposition 6.1.13. *The closed subsets given in Definition 6.1.12 are the closed subsets of a topology on* $\mathrm{pinj}_{\mathcal{T}}$ *which we will call the* ***$\mathcal{T}$-tensor-closed Ziegler topology*** *and denote by* $\mathrm{Zg}_{\mathcal{T}}^{\otimes}$.

Proof. Let $\{\mathcal{D}_i : i \in I\}$ be a family of $\mathcal{T}$-tensor-closed definable subcategories. Then $\bigcap_{i\in I}\mathcal{D}_i$ is also $\mathcal{T}$-tensor-closed as if $X \in \bigcap_{i\in I}\mathcal{D}_i$, $X \in \mathcal{D}_i$ for all $i \in I$. So for any $Y \in \mathcal{T}$, $X \otimes Y \in \mathcal{D}_i$ for all $i \in I$, which implies $X \otimes Y \in \bigcap_{i\in I}\mathcal{D}_i$. Therefore, if we set $\mathscr{C}_i := \mathcal{D}_i \cap \mathrm{pinj}_{\mathcal{T}}$, then the $\mathscr{C}_i$ are a collection of closed subsets of $\mathrm{pinj}_{\mathcal{T}}$ with respect to Definition 6.1.12. But we have

$$\bigcap_{i\in I}\mathscr{C}_i = \bigcap_{i\in I}(\mathcal{D}_i \cap \mathrm{pinj}_{\mathcal{T}}) = (\bigcap_{i\in I}\mathcal{D}_i) \cap \mathrm{pinj}_{\mathcal{T}},$$

so since $\bigcap_{i\in I}\mathcal{D}_i$ is $\mathcal{T}$-tensor-closed and definable, $\bigcap_{i\in I}\mathscr{C}_i$ is closed.

Now suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\mathcal{T}$-tensor-closed and definable, so $\mathscr{C}_i = \mathcal{D}_i \cap \mathrm{pinj}_{\mathcal{T}}$ are closed, for $i = 1,\ 2$. To show that $\mathscr{C}_1 \cup \mathscr{C}_2$ is closed we need to show that $\mathcal{D} := \left\langle \mathcal{D}_1 \cup \mathcal{D}_2 \right\rangle^{\mathrm{def}}$ is $\mathcal{T}$-tensor-closed. By [36], the Serre subcategory corresponding to $\mathcal{D}$ is $S_1 \cap S_2$, where S_i is the Serre subcategory associated to $\mathcal{D}_i$ for $i = 1,\ 2$. By Theorem 5.1.8, S_1 and S_2 are tensor-ideals of $\mathrm{Coh}(\mathcal{T})$. Therefore, $S_1 \cap S_2$ is also a tensor-ideal of $\mathrm{Coh}(\mathcal{T})$. Applying Theorem 5.1.8 again, we get that $\mathcal{D}$ is $\mathcal{T}$-tensor-closed, as required. $\square$

Remark 6.1.14. Note that every $\mathcal{T}$-tensor-closed definable subcategory is shift-closed as for all $X \in \mathcal{T}$, $\Sigma X \cong \Sigma 1 \otimes X$.

Our aim for the rest of this section is to answer the following question.

Question 6.1.15. Is there an equivalence relation $\sim_{\otimes}$ on $\mathrm{pinj}_{\mathcal{T}}$ such that $\mathrm{Zg}_{\mathcal{T}}^{\otimes}$ and $\mathrm{Zg}_{\mathcal{T}}/_{\sim_{\otimes}}$ have isomorphic frames of open subsets?

We will show that setting $P \sim_{\otimes} Q$ if and only if P and Q are topologically indistinguishable in $\mathrm{Zg}_{\mathcal{T}}^{\otimes}$ induces a split monomorphism in the category of frames $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\otimes}) \to \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}/_{\sim_{\otimes}})$, which in general is not an isomorphism.

Definition 6.1.16. Define an equivalence relation $\sim_{\otimes}$ on $\mathrm{pinj}_{\mathcal{T}}$ by $P \sim_{\otimes} Q$ if and only if they are topologically indistinguishable in $\mathrm{Zg}_{\mathcal{T}}^{\otimes}$.

Therefore we can consider the quotient topology $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$ with closed subsets $\mathscr{C} \subseteq \mathrm{pinj}_{\mathcal{T}}/_{\sim_\otimes}$ given by those subsets which satisfy $q^{-1}(\mathscr{C}) = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ for some definable subcategory $\mathcal{D} \subseteq \mathcal{T}$.

Let us provide an alternative characterisation of when two indecomposable pure-injectives, P and Q, are topologically indistinguishable in $\mathrm{Zg}^{\otimes}_{\mathcal{T}}$. Recall that for a full subcategory $\mathcal{X} \subseteq \mathcal{T}$, we denote by $\langle \mathcal{X} \rangle^{\mathrm{def}^{\otimes}}$ the smallest $\mathcal{T}$-tensor-closed definable subcategory containing $\mathcal{X}$.

Lemma 6.1.17. *P, $Q \in \mathrm{pinj}_{\mathcal{T}}$ are topologically indistinguishable in $\mathrm{Zg}^{\otimes}_{\mathcal{T}}$ if and only if $\langle P \rangle^{\mathrm{def}^{\otimes}} = \langle Q \rangle^{\mathrm{def}^{\otimes}}$.*

Proof. This is clear from the definition of $\mathrm{Zg}^{\otimes}_{\mathcal{T}}$. □

Now we show that the quotient map $q : \mathrm{pinj}_{\mathcal{T}} \to \mathrm{pinj}_{\mathcal{T}}/_{\sim_\otimes}$ maps closed subsets of $\mathrm{Zg}^{\otimes}_{\mathcal{T}}$ to closed subsets of $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$.

Lemma 6.1.18. *If $\mathcal{D} \subseteq \mathcal{T}$ is definable and $\mathcal{T}$-tensor-closed and $\mathscr{C} = q(\mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}})$, where $q : \mathrm{pinj}_{\mathcal{T}} \to \mathrm{pinj}_{\mathcal{T}}/_{\sim_\otimes}$ is the quotient map, then $q^{-1}(\mathscr{C}) = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$, in particular $\mathscr{C}$ is closed in $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$.*

Proof. If $P \in q^{-1}(\mathscr{C})$ then $P \sim_\otimes Q$ for some $Q \in \mathcal{D}$. Therefore, $\langle P \rangle^{\mathrm{def}^{\otimes}} = \langle Q \rangle^{\mathrm{def}^{\otimes}}$ and since $\mathcal{D}$ is $\mathcal{T}$-tensor-closed, $Q \in \mathcal{D}$ implies $\langle P \rangle^{\mathrm{def}^{\otimes}} = \langle Q \rangle^{\mathrm{def}^{\otimes}} \subseteq \mathcal{D}$ so $P \in \mathcal{D}$, as required. □

Theorem 6.1.19. *The quotient map $q : \mathrm{pinj}_{\mathcal{T}} \to \mathrm{pinj}_{\mathcal{T}}/_{\sim_\otimes}$ induces a split monomorphisms in the category of frames $\mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}}) \to \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes})$.*

Proof. First we show that for an open subset $\mathscr{O} \in \mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}})$, $q(\mathscr{O})$ is open in $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$. Indeed, the closed complement $\mathscr{C} = \mathrm{pinj}_{\mathcal{T}} \backslash \mathscr{O}$ of $\mathscr{O}$ is given by $\mathscr{C} = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ for some $\mathcal{T}$-tensor-closed definable subcategory $\mathcal{D}$. Therefore, by Lemma 6.1.18, $q(\mathscr{C})$ is closed with respect to the quotient topology and its inverse image is $\mathscr{C}$. Therefore, since q is onto, $q(\mathscr{O})$ is the open complement of $\mathscr{C}$.

Clearly the map $q : \mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}}) \to \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes})$ given by $\mathscr{O} \mapsto q(\mathscr{O})$ is inclusion-preserving and commutes with finite intersection and infinite union. Therefore, it is a morphism of frames.

We define a frame morphism $r : \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}) \to \mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}})$ by mapping an open subset $\mathscr{O}'$ of $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$ to the open complement of $\langle \mathcal{D} \rangle^{\mathrm{def}^\otimes} \cap \mathrm{pinj}_{\mathcal{T}}$ in $\mathrm{Zg}^{\otimes}_{\mathcal{T}}$, where $\mathcal{D}$ is the definable subcategory of $\mathcal{T}$ such that $q^{-1}(\mathscr{C}) = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ with $\mathscr{C} = (\mathrm{pinj}_{\mathcal{T}}/_{\sim_\otimes}) \backslash \mathscr{O}'$. Noting that

$$\langle \mathcal{D} \cup \mathcal{D}' \rangle^{\mathrm{def}^\otimes} = \langle \langle \mathcal{D} \rangle^{\mathrm{def}^\otimes} \cup \langle \mathcal{D}' \rangle^{\mathrm{def}^\otimes} \rangle^{\mathrm{def}}$$

and

$$\langle \bigcap_{i \in I} \mathcal{D}_i \rangle^{\mathrm{def}^\otimes} = \bigcap_{i \in I} \langle \mathcal{D}_i \rangle^{\mathrm{def}^\otimes},$$

it is easy to check that we have defined a morphism of frames. Finally note that $r \circ q = \mathrm{id}_{\mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}})}$ by Lemma 6.1.18. Therefore, q is a split monomorphism in the category of frames. $\square$

Example 6.1.20. *The split monomorphism in Theorem 6.1.19 is not in general an isomorphism. Indeed, let $\mathcal{T} = kV_4$-$\underline{\mathrm{Mod}}$ and identify the indecomposable pure-injectives of $\mathcal{T}$ with the string and band modules given in Example 2.6.21.*

In Example 5.1.15 we saw that $\langle M(a) \rangle^{\mathrm{def}}$ is $\mathcal{T}$-tensor-closed and in particular $M(a) \otimes M(ab^{-1}) \cong M(a)$ in kV_4-$\underline{\mathrm{Mod}}$. Let us consider the closure, $\mathscr{C}$, of $\{[M(ab^{-1})]/_{\sim_\otimes}\}$ in $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$. By definition of the quotient topology $q^{-1}(\mathscr{C}) = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ where $\mathcal{D}$ is a definable subcategory of $\mathcal{T}$. As $M(a)$ is finite dimensional, it is clopen in $\mathrm{Zg}_{\mathcal{T}}$. In addition $\langle M(a) \rangle^{\mathrm{def}} = \langle M(a) \rangle^{\mathrm{def}^\otimes}$, so $q^{-1}([M(a)]/_{\sim_\otimes}) = \{M(a)\}$ meaning $\{[M(a)]/_{\sim_\otimes}\}$ is clopen in $\mathrm{Zg}_{\mathcal{T}}/_{\sim_\otimes}$. Therefore $\{[M(a)]/_{\sim_\otimes}\} \notin \mathscr{C}$ as $\mathscr{C}$ is the smallest closed subset of $\mathrm{pinj}_{\mathcal{T}}/_{\sim_\otimes}$ containing $[M(ab^{-1})]/_{\sim_\otimes}$ and $\mathscr{C} \backslash \{[M(a)]/_{\sim_\otimes}\}$ is also closed.

Therefore, $M(ab^{-1}) \in \mathcal{D}$ but $M(a) \otimes M(ab^{-1}) \cong M(a) \notin \mathcal{D}$. Hence $\mathcal{D}$ is not $\mathcal{T}$-tensor-closed and the open complement of $\mathscr{C}$ is not in the image of the split embedding in Theorem 6.1.19.

6.1.3 Tensor-ideal Ziegler topology

Definition 6.1.21. We say that a subset $\mathscr{C} \subseteq \mathrm{pinj}_{\mathcal{T}}$ is **closed** with respect to the **tensor-ideal Ziegler topology** if there exists a definable tensor-ideal $\mathcal{D}$ of

$\mathcal{T}$ such that $\mathscr{C} = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$.

Let $\mathbb{D}(\mathcal{T})$, $\mathbb{D}^{\Sigma}(\mathcal{T})$, $\mathbb{D}^{\otimes}(\mathcal{T})$ and $\mathbb{D}^{\otimes\Delta}(\mathcal{T})$ denote the lattice of definable subcategories of $\mathcal{T}$, the lattice of shift-closed definable subcategories of $\mathcal{T}$, the lattice of $\mathcal{T}$-tensor-closed definable subcategories of $\mathcal{T}$ and the lattice of definable tensor-ideals of $\mathcal{T}$ respectively.

Corollary 6.1.22. *There exist isomorphisms of frames $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}) \cong \mathbb{D}(\mathcal{T})^{\mathrm{op}}$, $\mathbb{O}(\mathrm{Zg}^{\Sigma}_{\mathcal{T}}) \cong \mathbb{D}^{\Sigma}(\mathcal{T})^{\mathrm{op}}$ and $\mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}}) \cong \mathbb{D}^{\otimes}(\mathcal{T})^{\mathrm{op}}$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{O}(\mathrm{Zg}^{\otimes}_{\mathcal{T}}) \arrow[r, "\text{inclusion}"] \arrow[d, "\cong"'] & \mathbb{O}(\mathrm{Zg}^{\Sigma}_{\mathcal{T}}) \arrow[r, "\text{inclusion}"] \arrow[d, "\cong"] & \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}) \arrow[d, "\cong"] \\
\mathbb{D}^{\otimes}(\mathcal{T})^{\mathrm{op}} \arrow[r, "\text{inclusion}"] & \mathbb{D}^{\Sigma}(\mathcal{T})^{\mathrm{op}} \arrow[r, "\text{inclusion}"] & \mathbb{D}(\mathcal{T})^{\mathrm{op}}
\end{tikzcd}$$

In particular, $\mathbb{D}(\mathcal{T})$, $\mathbb{D}^{\Sigma}(\mathcal{T})$ and $\mathbb{D}^{\otimes}(\mathcal{T})$ are all dual frames.

Let us consider these different Ziegler-type topologies in the following simple example.

Example 6.1.23. *Let $\mathcal{T} = kG$-$\underline{\mathrm{Mod}}$ where k is a field of characteristic 5 and $G = \langle g \mid g^5 = 1 \rangle$ as in Example 2.6.17 (i). The following table shows the tensor product over k of these modules. The calculation was carried out using GAP (see [26]); see Appendix A.2 for the GAP code.*

$\otimes_k$	M_1	M_2	M_3	M_4	M_5
M_1	M_1	M_2	M_3	M_4	M_5
M_2	M_2	$M_1 \oplus M_3$	$M_2 \oplus M_4$	$M_3 \oplus M_5$	$M_5^{(2)}$
M_3	M_3	$M_2 \oplus M_4$	$M_1 \oplus M_3 \oplus M_5$	$M_2 \oplus M_5^{(2)}$	$M_5^{(3)}$
M_4	M_4	$M_3 \oplus M_5$	$M_2 \oplus M_5^{(2)}$	$M_1 \oplus M_5^{(3)}$	$M_5^{(4)}$
M_5	M_5	$M_5^{(2)}$	$M_5^{(3)}$	$M_5^{(4)}$	$M_5^{(5)}$

By Example 2.6.19, the Ziegler spectrum of $\mathcal{T} = kG$-$\underline{\mathrm{Mod}}$ is the discrete topology on four points. Note that $\Sigma M_i = M_{5-i}$ and $\Sigma^{-1} = \Sigma$. Therefore

$\mathrm{Zg}_{\mathcal{T}}^{\Sigma^+} = \mathrm{Zg}_{\mathcal{T}}^{\Sigma^-} = \mathrm{Zg}_{\mathcal{T}}^{\Sigma}$ *are all the same topology on* $\mathrm{pinj}_{\mathcal{T}} = \{M_1, M_2, M_3, M_4\}$ *with open subsets*

$$\{\emptyset, \{M_1, M_4\}, \{M_2, M_3\}, \mathrm{pinj}_{\mathcal{T}}\}.$$

Clearly the only $\mathcal{T}$*-tensor-closed definable subcategories of* kG-Mod *are* 0, $\langle M_5 \rangle^{\mathrm{def}}$ *and* kG-Mod. *Thus the* $\mathcal{T}$*-tensor-closed Ziegler topology of* kG-$\underline{\mathrm{Mod}}$ *is the trivial topology on four points.*

The next example shows that $\mathbb{D}^{\otimes}(\mathcal{T})$ and $\mathbb{D}^{\otimes\Delta}(\mathcal{T})$ are in general different lattices.

Example 6.1.24. *Let* $\mathcal{T} = kV_4$-$\underline{\mathrm{Mod}}$ *and identify the indecomposable pure-injectives of* $\mathcal{T}$ *with the string and band modules given in Example 2.6.21. In Example 5.1.15 we saw that* $\langle M(a) \rangle^{\mathrm{def}}$ *is* $\mathcal{T}$*-tensor-closed. We show that* $\langle M(a) \rangle^{\mathrm{def}}$ *is not closed under extensions. Indeed, we can define a short exact sequence* $0 \to M(a) \xrightarrow{\alpha} M(ab^{-1}a) \xrightarrow{\beta} M(a) \to 0$ *in* kV_4-Mod *as follows. Denote the generators of* $M(a)$ *by* x_0 *and* x_1 *with the action of* a *sending* x_0 *to* x_1*. This can be pictured as follows.*

$$\begin{tikzcd} & x_0 \arrow[dl, "a"'] \\ x_1 & \end{tikzcd}$$

Denote the generators of $M(ab^{-1}a)$ *by* x_0, x_1, x_2, x_3 *with the action of* a *and* b *as pictured below.*

$$\begin{tikzcd} & x_2 \arrow[dl, "a"'] \arrow[dr, "b"] & & x_0 \arrow[dl, "a"] \\ x_3 & & x_1 & \end{tikzcd}$$

Define $\alpha : M(a) \to M(ab^{-1}a)$ *by mapping* $x_0 \mapsto x_0$ *and* $x_1 \mapsto x_1$ *and define* $\beta : M(ab^{-1}a) \to M(a)$ *by mapping* $x_0, x_1 \mapsto 0$, $x_2 \mapsto x_0$ *and* $x_3 \mapsto x_1$*. Therefore we have an exact triangle* $M(a) \xrightarrow{\alpha} M(ab^{-1}a) \xrightarrow{\beta} M(a) \to \Sigma M(a)$ *in* kV_4-$\underline{\mathrm{Mod}}$ *and* $\langle M(a) \rangle^{\mathrm{def}}$ *is* $\mathcal{T}$*-tensor-closed but not a tensor-ideal.*

6.2 The Balmer spectrum of $\mathcal{T}^c$

Remark 6.2.1. In [11, Theorem 5.5] it is shown that the lattice of smashing tensor-ideals, $\mathbb{S}^{\otimes}(\mathcal{T})$, of a rigidly-compactly generated tensor triangulated category, $\mathcal{T}$, is complete and forms a frame.

By Theorem 2.6.11, the lattice, $\mathrm{Thom}(\mathcal{T}^c)$, of Thomason subsets of $\mathrm{Spc}(\mathcal{T}^c)$ is isomorphic to the lattice, $\mathrm{Thick}^{\otimes}(\mathcal{T}^c)$, of thick tensor-ideals of $\mathcal{T}^c$.

Next we define what it means for the Telescope Conjecture to hold for $\mathcal{T}$. As we are working in the tensor triangulated setting, we provide both a non-tensor and a tensor version of this conjecture.

Definition 6.2.2. (i) Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category. We say that the **Telescope Conjecture** holds for $\mathcal{T}$ if every smashing subcategory $\mathcal{B} \subseteq \mathcal{T}$ is generated as a localising subcategory by some thick subcategory $I \subseteq \mathcal{T}^c$. In this case, we necessarily have $I = \mathcal{B} \cap \mathcal{T}^c$.

(ii) Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category. We say that the **tensor-Telescope Conjecture** holds for $\mathcal{T}$ if every smashing tensor-ideal $\mathcal{B} \subseteq \mathcal{T}$ is generated as a localising subcategory by some thick tensor-ideal $I \subseteq \mathcal{T}^c$. In this case, we necessarily have $I = \mathcal{B} \cap \mathcal{T}^c$, (e.g. see [10, Definition 4.2])

Remark 6.2.3. By Theorem 5.2.10, the Telescope Conjecture holds for $\mathcal{T}$ if and only if every triangulated definable subcategory of $\mathcal{T}$ has form

$$\{X \in \mathcal{T} : (A, X) = 0, \forall A \in I\},$$

for some thick tensor-ideal $I \subseteq \mathcal{T}^c$.

Proposition 6.2.4. *[10, Definition 6.1 and Proposition 6.2] There exists an injective order-preserving map*

$$\Delta : \mathrm{Thom}(\mathcal{T}^c) \hookrightarrow \mathbb{S}^{\otimes}(\mathcal{T})$$

which is a lattice isomorphism if and only if the tensor-Telescope Conjecture holds for $\mathcal{T}$.

Proof. (sketch) Δ is given by the composition

$$\mathrm{Thom}(\mathcal{T}^c) \cong \mathrm{Thick}^{\otimes}(\mathcal{T}^c) \hookrightarrow \mathbb{S}^{\otimes}(\mathcal{T}),$$

where $\mathrm{Thick}^{\otimes}(\mathcal{T}^c) \hookrightarrow \mathbb{S}^{\otimes}(\mathcal{T})$ maps a thick tensor-ideal of $\mathcal{T}^c$ to the localising subcategory of $\mathcal{T}$ it generates. By [10, Theorem 4.1(a)], the localising subcategory of $\mathcal{T}$ generated by a thick tensor-ideal of $\mathcal{T}^c$ is a smashing tensor-ideal.

We call the map $\mathrm{Thick}^{\otimes}(\mathcal{T}^c) \hookrightarrow \mathbb{S}^{\otimes}(\mathcal{T})$ **inflation**. It is always injective and is surjective if and only if the tensor-Telescope Conjecture holds for $\mathcal{T}$ [10, Proposition 6.2]. Therefore, if the tensor-Telescope Conjecture holds for $\mathcal{T}$ then inflation is a lattice isomorphism $\mathrm{Thick}^{\otimes}(\mathcal{T}^c) \cong \mathbb{S}^{\otimes}(\mathcal{T})$. □

Let us consider an example. First we recall a useful lemma.

Lemma 6.2.5. *[57, Lemma 1.9] Let $\mathcal{K}$ be a skeletally small rigid tensor triangulated category. If $\mathcal{K}$ is the smallest thick subcategory containing the tensor unit,* 1, *then every thick subcategory of $\mathcal{K}$ is a thick tensor-ideal.*

Example 6.2.6. *Suppose R is a commutative noetherian ring. Then by ([8, Theorem 6.3(a)] and [9, Example 4.4]) the Balmer spectrum,* $\mathrm{Spc}(D^c(\mathrm{Mod}\text{-}R))$, *is homeomorphic to the Zariski spectrum,* $\mathrm{Spec}(R)$. *In addition, by [45, Corollary 3.4], the Telescope Conjecture holds for $\mathcal{T} = D(\mathrm{Mod}\text{-}R)$ and the thick subcategories of $D^c(\mathrm{Mod}\text{-}R)$ correspond to the specialisation closed subsets of* $\mathrm{Spec}(R)$. *By Lemma 6.2.5, every thick subcategory is a thick tensor-ideal and therefore we have a bijection between $\mathbb{D}^{\otimes\Delta}(\mathcal{T})$ and the specialisation closed subsets of the Zariski spectrum* $\mathrm{Spec}(R)$.

In the remainder of this section we will consider the case where $\mathcal{T}$ is the stable module category of a group algebra.

Definition 6.2.7. Let G be a finite group, k be an algebraically closed field of characteristic $p > 0$ and M be a finite dimensional kG-module. Let $H^i(G, M) = \mathrm{Ext}^i_{kG}(k, M)$ where $\mathrm{Ext}^i_{kG}(k, M)$ is the ith cohomology of

$$0 \to \mathrm{Hom}_{kG}(P^0, M) \to \mathrm{Hom}_{kG}(P^1, M) \to \mathrm{Hom}_{kG}(P^2, M) \to ...$$

where $\ldots \to P^2 \xrightarrow{p^2} P^1 \xrightarrow{p^1} P^0 \xrightarrow{p^0} k \to 0$ is a projective resolution of k as a kG-module.

Set $H^\bullet(G,k) = \bigoplus_{i\in\mathbb{Z}} H^i(G,k)$. This is a finitely generated graded-commutative k-algebra.

Definition 6.2.8. Let $\mathrm{Proj}(H^\bullet(G,k))$ denote the space of maximal homogeneous ideals of the graded-commutative algebra $H^\bullet(G,k)$ strictly contained in the maximal ideal of positive degree elements. We endow $\mathrm{Proj}(H^\bullet(G,k))$ with the Zariski topology. That is the closed subsets of $\mathrm{Proj}(H^\bullet(G,k))$ are given by

$$\{V \in \mathrm{Proj}(H^\bullet(G,k)) : I \subseteq V\},$$

as I varies over the homogeneous ideals of $H^\bullet(G,k)$.

For any $M \in kG\text{-}\underline{\mathrm{mod}}$ we introduce a support variety, $V_G(M)$ in $\mathrm{Proj}(H^\bullet(G,k))$.

Definition 6.2.9. For any $M \in kG\text{-}\underline{\mathrm{mod}}$, define $I_G(M)$ to be the homogeneous ideal given by the kernel of the map

$$H^\bullet(G,k) = \mathrm{Ext}^\bullet_{kG}(k,k) \xrightarrow{-\otimes_k M} \mathrm{Ext}^\bullet_{kG}(M,M).$$

We then set

$$V_G(M) = \{V \in \mathrm{Proj}(H^\bullet(G,k)) : I_G(M) \subseteq V\}.$$

Proposition 6.2.10. *([15, Proposition 3.3 and Theorem 3.4] and [8, Theorem 5.9])* $(\mathrm{Proj}(H^\bullet(G,k)), \sigma)$ *is a classifying support data on* kG-$\underline{\mathrm{mod}}$*, where* σ *is given by* $M \mapsto V_G(M)$.

Remark 6.2.11. In [15, Proposition 3.3 and Theorem 3.4] it is shown that the thick tensor-ideals of kG-$\underline{\mathrm{mod}}$ correspond to non-empty sets of closed homogeneous subvarieties $\mathcal{X}$ of $\mathrm{Proj}(H^\bullet(G,k))$ which are closed under specialization in the sense that if $W \in \mathcal{X}$ and $W' \subseteq W$ then $W' \in \mathcal{X}$. Notice that $\mathcal{X} \mapsto \bigcup_{W\in\mathcal{X}} W$ gives a one-to-one correspondence between these sets and the specialization closed subsets of the topology $\mathrm{Proj}(H^\bullet(G,k))$.

Using Theorem 2.6.8 we get the following corollary.

Corollary 6.2.12. *There exists a homeomorphism*

$$\mathrm{Spc}(kG\text{-}\underline{\mathrm{mod}}) \cong \mathrm{Proj}(H^\bullet(G,k)).$$

Proposition 6.2.13. *[16, Theorem 11.12] Let $\mathcal{T} = kG$-$\underline{\mathrm{Mod}}$ where G is a finite group and k is a field of characteristic p where p divides the order of the group. Then the tensor-Telescope Conjecture holds for $\mathcal{T}$.*

Thus, by Theorem 5.2.13, the thick tensor-ideals of kG-$\underline{\mathrm{mod}}$ correspond via an inclusion-reversing map to the definable tensor-ideals of kG-$\underline{\mathrm{Mod}}$. Let us consider a particular example.

Example 6.2.14. *Let $G = V_4$ be the Klein four group, that is $V_4 = \langle x, y \mid x^2 = y^2 = [x,y] = e_G \rangle \cong C_2 \times C_2$, k be an algebraically closed field of order 2 and set $\mathcal{T} = kV_4$-$\underline{\mathrm{Mod}}$. By [17, Section 4.3], the non-trivial proper thick tensor-ideals of kV_4-$\underline{\mathrm{mod}}$ are indexed by the projective line over k. In particular, $\{M({}^n(b^{-1}a)b^{-1}) : n \in \mathbb{Z}^{\geq 0}\}$ and $\{M({}^n(ab^{-1})a) : n \in \mathbb{Z}^{\geq 0}\}$ are the indecomposable modules contained in two thick tensor-ideals and for each $\lambda \in k^\times$ the set of band modules $\{B(ab^{-1}, \lambda, n) : n \in \mathbb{N}\}$ is the set of indecomposable modules contained in a thick tensor-ideal of kV_4-$\underline{\mathrm{mod}}$. As the (tensor version of the) telescope conjecture holds for $\mathcal{T}$, the definable tensor-ideals of kV_4-$\underline{\mathrm{Mod}}$ have the form $I^\perp$ where I is a thick tensor-ideal of kV_4-$\underline{\mathrm{mod}}$. Thus the non-trivial proper definable tensor-ideals of $\mathcal{T} = kV_4$-$\underline{\mathrm{Mod}}$ are indexed by the projective line over k.*

Chapter 7

Internal tensor-duality

Fix a rigidly-compactly generated tensor triangulated category $\mathcal{T}$. In this section we define an internal tensor-duality of definable subcategories of $\mathcal{T}$.

7.1 Defining internal tensor-duality

For any $f : A \to B$ in $\mathcal{T}^c$ there exists a dual morphism $f^\vee : B^\vee \to A^\vee$ given by the following composition,

$$B^\vee \xrightarrow{\eta_A \otimes B^\vee} A^\vee \otimes A \otimes B^\vee \xrightarrow{A^\vee \otimes f \otimes B^\vee} A^\vee \otimes B \otimes B^\vee \xrightarrow{A^\vee \otimes \varepsilon_B} A^\vee.$$

Furthermore, for any $C \in \mathcal{T}^c$ there exist an isomorphism $\delta_C : C \to (C^\vee)^\vee$ such that for any $f : A \to B$ in $\mathcal{T}^c$ we have $f = \delta_B^{-1} \circ (f^\vee)^\vee \circ \delta_A$. Therefore, given any ideal of morphisms $\mathcal{J}$, $f \in \mathcal{J}$ if and only if $(f^\vee)^\vee \in \mathcal{J}$.

Definition 7.1.1. Given a cohomological ideal $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$, we will call the set $\mathcal{J}^\vee = \{f^\vee : f \in \mathcal{J}\}$, the **internal tensor-dual** of $\mathcal{J}$.

By the above discussion, $(\mathcal{J}^\vee)^\vee = \mathcal{J}$. We will show that the internal tensor-dual of $\mathcal{J}$ is also a cohomological ideal, and use the assignment $\mathcal{J} \mapsto \mathcal{J}^\vee$ to define an internal tensor-duality of definable subcategories.

We apply the following Theorem.

Theorem 7.1.2. *[27, Theorem 7.4] Let $\mathcal{K}$ and $\mathcal{T}$ be compactly generated triangulated categories. Suppose there is a duality between $\mathcal{K}^c$ and $\mathcal{T}^c$ and denote by $\Theta : \text{mod-}\mathcal{K}^c \xrightarrow{\sim} \mathcal{T}^c\text{-mod}$ the induced equivalence of categories. Let $\Delta : \mathcal{T}^c\text{-mod} \to \text{mod-}\mathcal{T}^c$ be the duality defined by $\Delta M(X) = (M, H^X)$ for all $X \in \mathcal{T}^c$, where $H^X = (X, -)|_{\mathcal{T}^c}$ and set $\Gamma = \Delta \circ \Theta : \text{mod-}\mathcal{K}^c \to \text{mod-}\mathcal{T}^c$.*

Then Γ yields an inclusion-preserving bijective correspondence between the Serre subcategories of $\text{mod-}\mathcal{K}^c$ and the Serre subcategories of $\text{mod-}\mathcal{T}^c$ and therefore induces an isomorphism between the open subsets of $\text{Zg}_{\mathcal{K}}$ and $\text{Zg}_{\mathcal{T}}$.

Since $(-)^\vee : \mathcal{T}^c \to \mathcal{T}^c$ is a duality, we have an inclusion-preserving bijective correspondence Γ between the Serre subcategories of $\text{mod-}\mathcal{T}^c$ and $\text{mod-}\mathcal{T}^c$ which induces an automorphism on the opens subsets of the Ziegler spectrum, $\text{Zg}_{\mathcal{T}}$. We show that this bijective correspondence coincides with the assignment $\mathcal{J} \mapsto \mathcal{J}^\vee$ on the related (by Theorem 2.5.11) cohomological ideals.

Given $(-, A) \xrightarrow{(-,f)} (-, B) \to G_f \to 0$ in $\text{mod-}\mathcal{T}^c$, $\Theta(G_f) = F_{f^\vee} \in \mathcal{T}^c\text{-mod}$ has presentation $(A^\vee, -) \xrightarrow{(f^\vee,-)} (B^\vee, -) \to F_{f^\vee} \to 0$. Note that this is the duality defined in Section 4.4, with $\mathcal{A} = \mathcal{T}^c$.

Suppose $A \xrightarrow{f} B \xrightarrow{f'} C \xrightarrow{f''} \Sigma A$ is an exact triangle in $\mathcal{T}^c$. Then, since $(-)^\vee$ is exact we have an exact triangle $(\Sigma A)^\vee \xrightarrow{f''^\vee} C^\vee \xrightarrow{f'^\vee} B^\vee \xrightarrow{f^\vee} A^\vee$ and by definition of Δ, we have $\Gamma(G_f) = \Delta(F_{f^\vee}) = G_{f''^\vee}$.

Suppose $A \xrightarrow{f} B \xrightarrow{f'} C \xrightarrow{f''} \Sigma A$ is an exact triangle in $\mathcal{T}^c$ and therefore so is $(\Sigma A)^\vee \xrightarrow{f''^\vee} C^\vee \xrightarrow{f'^\vee} B^\vee \xrightarrow{f^\vee} A^\vee$. If $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ is a Serre subcategory with corresponding cohomological ideal $\mathcal{J}$, then by Lemma 5.1.4, $G_f \in \mathsf{C}$ if and only if $f' \in \mathcal{J}$. By Theorem 7.1.2, $\Gamma\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ is also a Serre subcategory and by the above $G_f \in \mathsf{C}$ if and only if $G_{f''^\vee} \in \Gamma\mathsf{C}$. Therefore, applying Lemma 5.1.4 again, $f' \in \mathcal{J}$ if and only if $f'^\vee$ is in the cohomological ideal associated to $\Gamma\mathsf{C}$. In other words, if the cohomological ideal associated to C is $\mathcal{J}$ then the cohomological ideal associated to $\Gamma\mathsf{C}$ is $\mathcal{J}^\vee$.

Definition 7.1.3. Given a definable subcategory $\mathcal{D} \subseteq \mathcal{T}$ associated to the cohomological ideal $\mathcal{J}$, we denote by $\mathcal{D}^\vee$ the definable subcategory of $\mathcal{T}$ associated to $\mathcal{J}^\vee$. We call $\mathcal{D}^\vee$ the **internal tensor-dual** of $\mathcal{D}$.

Given a Serre subcategory $\mathsf{S} \subseteq \text{Coh}(\mathcal{T})$ associated to the cohomological ideal

$\mathcal{J}$ we denote by $\mathsf{S}^\vee$ the Serre subcategory of $\mathrm{Coh}(\mathcal{T})$ associated to $\mathcal{J}^\vee$. We call $\mathsf{S}^\vee$ the **internal tensor-dual** of S.

Given a Serre subcategory $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ where $\mathsf{S} = \delta\mathsf{C} \subseteq \mathrm{Coh}(\mathcal{T})$, we denote by $\mathsf{C}^\vee \subseteq \text{mod-}\mathcal{T}^c$ the Serre subcategory $\delta\mathsf{S}^\vee$. We call $\mathsf{C}^\vee$ the **internal tensor-dual** of C.

Remark 7.1.4. Recall that every pp formula in the language $\mathscr{L}(\mathcal{T})$ is equivalent to a division formula ϕ_f of the form $\exists y_B, x_A = y_B f$ for some $f : A \to B$ in $\mathcal{T}^c$ (see Section 2.5). Therefore we can define an internal tensor-duality of pp formulas by $\phi_f \mapsto \phi_{f^\vee}$. Notice here that if ϕ_f and $\phi_{f'}$ are equivalent then $\phi_{f^\vee}$ and $\phi_{f'^\vee}$ may not be equivalent pp formulas but they will be isomorphic. Indeed if $f : A \to B$ and $f' : A \to B'$, then $\phi_{f^\vee}$ and $\phi_{f'^\vee}$ have free variable of sort $B^\vee$ and $B'^\vee$ respectively and therefore are not equivalent if $B \neq B'$. However, by Proposition 2.5.4, there exist morphisms $k : B \to B'$ and $l : B' \to B$ such that $f = l \circ f'$ and $f' = k \circ f$. For any $X \in \mathcal{T}$, $- \circ k^\vee : \phi_{f^\vee}(X) \to \phi_{f'^\vee}(X)$ and $- \circ l^\vee : \phi_{f'^\vee}(X) \to \phi_{f^\vee}(X)$ define inverse group isomorphisms, i.e. $\phi_{f^\vee}$ and $\phi_{f'^\vee}$ are naturally isomorphic when regarded as coherent functors $\mathcal{T} \to \mathbf{Ab}$. Indeed, in order to define a specific functor $\Gamma : \text{mod-}\mathcal{T}^c \to \text{mod-}\mathcal{T}^c$ one needs to fix a choice for the presentation of each finitely presented functor in $\text{mod-}\mathcal{T}^c$. However, since the choices for Γ related to each selection are naturally isomorphic, all choices give rise to the same duality on definable subcategories.

7.2 Properties

In this section we explore some properties of internal tensor-duality.

Lemma 7.2.1. *Given any set of morphisms $I \subseteq \mathrm{morph}(\mathcal{T}^c)$,*

$$\langle I^\vee \rangle^{\mathrm{cohom}} = (\langle I \rangle^{\mathrm{cohom}})^\vee.$$

Proof. By the discussion after Theorem 7.1.2, we have seen that $(\langle I^\vee \rangle^{\mathrm{cohom}})^\vee$ is a cohomological ideal. Therefore, since $\langle I^\vee \rangle^{\mathrm{cohom}}$ contains $I^\vee$, $(\langle I^\vee \rangle^{\mathrm{cohom}})^\vee$ contains I and we must have $\langle I \rangle^{\mathrm{cohom}} \subseteq (\langle I^\vee \rangle^{\mathrm{cohom}})^\vee$. Applying $(-)^\vee$ we get $(\langle I \rangle^{\mathrm{cohom}})^\vee \subseteq$

$\langle I^\vee \rangle^{\text{cohom}}$. For the converse note that $I^\vee \subseteq (\langle I \rangle^{\text{cohom}})^\vee$ and $(\langle I \rangle^{\text{cohom}})^\vee$ is a cohomological ideal so $\langle I^\vee \rangle^{\text{cohom}} \subseteq (\langle I \rangle^{\text{cohom}})^\vee$. Hence $\langle I^\vee \rangle^{\text{cohom}} = (\langle I \rangle^{\text{cohom}})^\vee$ as required. $\square$

Proposition 7.2.2. *Let $\mathcal{D} \subseteq \mathcal{T}$ be a definable subcategory with corresponding cohomological ideal $\mathcal{J}$. If $\mathcal{D}$ is $\mathcal{T}$-tensor-closed then $\mathcal{D} = \mathcal{D}^\vee$.*

Proof. By Theorem 5.1.8, $\mathcal{D}$ is a $\mathcal{T}$-tensor-closed if and only if it is $\mathcal{T}^c$-tensor-closed if and only if $\mathcal{J}$ is $\mathcal{T}^c$-tensor-closed.

Recall that for $f : A \to B$ in $\mathcal{T}^c$, $f^\vee$ is given by

$$(A^\vee \otimes \varepsilon_B) \circ (A^\vee \otimes f \otimes B^\vee) \circ (\eta_A \otimes B^\vee).$$

Therefore, as $\mathcal{J}$ is $\mathcal{T}^c$-tensor-closed, $A^\vee \otimes f \otimes B^\vee \in \mathcal{J}$ so $f^\vee \in \mathcal{J}$. So $\mathcal{J}^\vee \subseteq \mathcal{J}$. Consequently, $\mathcal{J} = (\mathcal{J}^\vee)^\vee \subseteq \mathcal{J}^\vee$, giving equality as required. $\square$

Remark 7.2.3. The converse to Proposition 7.2.2 does not hold. Indeed, in Example 7.3.1 below, the definable subcategory generated by $\{M_1, M_4\}$ is self-dual with respect to internal tensor-duality, however it is not $\mathcal{T}$-tensor-closed (consider the table in Example 6.1.23).

Proposition 7.2.4. *Suppose $\mathcal{D} \subseteq \mathcal{T}$ is a positive shift-closed definable subcategory, that is $\Sigma\mathcal{D} \subseteq \mathcal{D}$. Then the internal tensor-dual $\mathcal{D}^\vee \subseteq \mathcal{T}$ is a negative shift-closed definable subcategory.*

Proof. Suppose $\mathcal{D} \subseteq \mathcal{T}$ is a positive shift-closed definable subcategory with associated cohomological ideal $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$. Then $\Sigma^{-1}\mathcal{J} \subseteq \mathcal{J}$ by Corollary 6.1.3 and so $(\Sigma^{-1}\mathcal{J})^\vee \subseteq \mathcal{J}^\vee$. Recall that $(-)^\vee$ is exact, so in particular for any $f \in \mathrm{morph}(\mathcal{T}^c)$, $(\Sigma^{-1} f)^\vee \cong \Sigma f^\vee$ so $\Sigma\mathcal{J}^\vee \subseteq \mathcal{J}^\vee$. Consequently, if $X \in \mathcal{D}^\vee$, then for all $g \in \mathcal{J}^\vee$, $(\Sigma g, X) = 0 = (g, \Sigma^{-1} X)$. Therefore, $\Sigma^{-1} X \in \mathcal{D}^\vee$. Hence $\mathcal{D}^\vee$ is negative shift-closed as required. $\square$

Theorem 7.2.5. *Internal tensor-duality induces a lattice automorphism on $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}})$ which gives an isomorphism $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^+}) \cong \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^-})$, restricts to an automorphism on $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma}) = \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^+}) \cap \mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\Sigma^-})$ and fixes $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}}^{\otimes})$.*

Proof. First we show that the induced map is inclusion-preserving. Given open subsets $\mathscr{O} \subseteq \mathscr{O}'$ in $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}})$, the closed complements satisfy $\mathrm{pinj}_{\mathcal{T}} \backslash \mathscr{O} = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ and $\mathrm{pinj}_{\mathcal{T}} \backslash \mathscr{O}' = \mathcal{D}' \cap \mathrm{pinj}_{\mathcal{T}}$ where $\mathcal{D}$ and $\mathcal{D}'$ are definable subcategories with $\mathcal{D}' \subseteq \mathcal{D}$. Therefore the corresponding cohomological ideals satisfy $\mathcal{J}_{\mathcal{D}} \subseteq \mathcal{J}_{\mathcal{D}'}$ and since $(-)^\vee$ is inclusion-preserving we have $\mathcal{J}_{\mathcal{D}}^\vee \subseteq \mathcal{J}_{\mathcal{D}'}^\vee$. So $\mathcal{D}'^\vee \subseteq \mathcal{D}^\vee$ which gives $\mathscr{O}^\vee \subseteq \mathscr{O}'^\vee$ where $\mathscr{O}^\vee := \mathrm{pinj}_{\mathcal{T}} \backslash (\mathcal{D}^\vee \cap \mathrm{pinj}_{\mathcal{T}})$ and $\mathscr{O}'^\vee := \mathrm{pinj}_{\mathcal{T}} \backslash (\mathcal{D}'^\vee \cap \mathrm{pinj}_{\mathcal{T}})$. Therefore since $(-)^\vee$ is clearly self-inverse, we have an automorphism on $\mathbb{O}(\mathrm{Zg}_{\mathcal{T}})$. It remains to apply Proposition 7.2.2 and Proposition 7.2.4. $\square$

7.3 Examples

Let us consider some examples.

Example 7.3.1. *Suppose $G = \langle g \mid g^5 = 1 \rangle$ is the cyclic group of order five and let k be a field of characteristic 5. Then $kG \cong k[T]/(T^5)$ under the isomorphism $T \mapsto g - 1$. As in Example 2.6.19, let $M_i = k[T]/(T^i)$ for $i = 1, ..., 5$ be the indecomposable (finite dimensional) modules.*

Denote by $\phi_{ij} : M_n \to M_m$ the k-linear map which takes T^{i-1} to T^{j-1}. Then the ϕ_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq m$ form a basis for all k-linear maps from M_n to M_m.

Let $f = \phi_{12} + \phi_{23} : M_2 \to M_3$ and consider the definable subcategory $\mathcal{D} = \{X \in kG\text{-}\underline{\mathrm{Mod}} : \underline{\mathrm{Hom}}(f, X) = 0\}$. We claim that $\mathcal{D} = \langle M_1 \rangle^{\mathrm{def}}$. Since every definable subcategory is generated by its indecomposable pure-injectives, it is sufficient to check that $M_1 \in \mathcal{D}$ but $M_2, ..., M_4 \notin \mathcal{D}$.

*Since there are no non-zero kG-linear morphisms from $M_2 \to M_5 \to M_3$, f is non-zero in $\underline{\mathrm{Hom}}(M_2, M_3)$. Also note that the only kG-linear morphisms from $M_3 \to M_1$ have the form $\lambda\phi_{11}$ for some $\lambda \in k$. Therefore for any $h = \lambda\phi_{11} : M_3 \to M_1$, $h \circ f = 0$, in other words $(f, M_1) = 0$ (both in kG-*Mod *and kG-*$\underline{\text{Mod}}$*) and $M_1 \in \mathcal{D}$.*

$M_2 \notin \mathcal{D}$ as $\lambda(\phi_{11} + \phi_{22}) : M_3 \to M_2$ is kG-linear for any $\lambda \in k$, and $\lambda(\phi_{11} + \phi_{22}) \circ f = \lambda\phi_{12} : M_2 \to M_2$ is non-zero when λ is non-zero. $M_3 \notin \mathcal{D}$, as (f, M_3) :

$(M_3, M_3) \to (M_2, M_3)$ *maps* $\mathrm{id}_{M_3} \mapsto f$ *and* $M_4 \notin \mathcal{D}$ *as* $(\phi_{12} + \phi_{23} + \phi_{34}) \circ f = \phi_{13} + \phi_{24}$ *which is non-zero in* $\underline{\mathrm{Hom}}(M_2, M_4)$.

Now let us calculate the dual definable subcategory of $\mathcal{D}$. *Let* $X \in kG\text{-mod}$ *have fixed* k*-basis,* $\{x_1, ..., x_n\}$. *The unit,* $\eta_X : k \to X^\vee \otimes X$, *and counit,* $\varepsilon_X : X \otimes X^\vee \to k$, *of the adjunction* $X \otimes - \dashv X^\vee \otimes -$ *are given by* $\eta_X : 1 \mapsto \sum_{j=1}^n \phi_{j1} \otimes x_j$ *and ,* $\varepsilon_X : x_i \otimes \phi_{j1} \mapsto \delta_{ij}$, *where* $\phi_{j1} : X \to k$ *maps* $x_j \to 1$.

Similarly the unit and counit of the adjunction $X^\vee \otimes - \dashv X \otimes -$ *are given by* $\eta'_X : k \to X \otimes X^\vee$, $1 \mapsto \sum_{j=1}^n x_j \otimes \phi_{j1}$ *and* $\varepsilon'_X : X^\vee \otimes X \to k$, $\phi_{j1} \otimes x_i \mapsto \delta_{ji}$.

Recall that given a morphism $g : A \to B$ *in* $kG\text{-mod}$, $g^\vee = (A^\vee \otimes \varepsilon_B) \circ (A^\vee \otimes g \otimes B^\vee) \circ (\eta_A \otimes B^\vee)$. *In particular if we fix* $\{1, T\}$ *as an ordered basis for* M_2 *and* $\{1, T, T^2\}$ *as an ordered basis for* M_3, *we have*

$$f^\vee : \phi_{i1} \mapsto \sum_{j=1}^{2} \phi_{j1} \otimes T^{j-1} \otimes \phi_{i1} \mapsto \sum_{j=1}^{2} \phi_{j1} \otimes T^{j} \otimes \phi_{i1} \mapsto \phi_{i-11},$$

where $\phi_{01} = 0$.

It is straight forward to check that for any $\phi_{i1} \in M_n^\vee$,

$$T\phi_{i1} = \begin{cases} 0, & \text{if } i = 1 \\ (-1)^i \sum_{j=1}^{i-1} (-1)^j \phi_{j1}, & i \geq 2. \end{cases}$$

Therefore, we can define an isomorphism $M_3 \cong M_3^\vee$ *given by* $T^2 \mapsto \phi_{11}$, $T \mapsto -\phi_{21}$ *and* $1 \mapsto \phi_{31} + \phi_{21}$. *Similarly, we have an isomorphism* $M_2^\vee \cong M_2$ *given by* $\phi_{11} \mapsto T$, $-\phi_{21} \mapsto 1$. *Thus,* $f^\vee \cong \phi_{12} - \phi_{11} - \phi_{22} : M_3 \to M_2$.

$M_1 \notin \mathcal{D}^\vee$ *as* $\phi_{11} \circ (\phi_{12} - \phi_{11} - \phi_{22}) = -\phi_{11} : M_3 \to M_1$ *is non-zero in* $\underline{\mathrm{Hom}}(M_3, M_1)$. *Clearly,* $M_2 \notin \mathcal{D}^\vee$. *In addition,* $(\phi_{12} + \phi_{23}) \circ (\phi_{12} - \phi_{11} - \phi_{22}) = -\phi_{12} - \phi_{23} + \phi_{13} : M_3 \to M_3$ *is non-zero in* $\underline{\mathrm{Hom}}(M_3, M_1)$ *so* $M_3 \notin \mathcal{D}^\vee$. *Finally, the only* kG*-linear maps from* $M_2 \to M_4$ *have form* $a(\phi_{13} + \phi_{24}) + b\phi_{14}$, *so any map in the image of* $(\phi_{12} - \phi_{11} - \phi_{22}, M_4)$ *has form* $a(-\phi_{13} - \phi_{24} + \phi_{14}) - b\phi_{14}$ *for some* a, $b \in k$. *However, this map factors via the projective* M_5 *as*

$$M_3 \xrightarrow{-(\phi_{13}+\phi_{24}+\phi_{35})+\phi_{14}+\phi_{25}} M_5 \xrightarrow{a(\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44})+b(\phi_{12}+\phi_{23}+\phi_{34})} M_4,$$

so $M_4 \in \mathcal{D}^\vee$.

In summary, if $\mathcal{D} = \langle M_1 \rangle^{\text{def}}$ *then* $\mathcal{D}^\vee = \langle M_4 \rangle^{\text{def}}$.

Now let $g = \phi_{11} : M_3 \to M_1$ *and* $h = \phi_{11} : M_4 \to M_1$. *Let* $\mathcal{J}_g$ *and* $\mathcal{J}_h$ *be the cohomological ideals generated by* g *and* h *respectively and denote the corresponding definable subcategories by* $\mathcal{D}_g$ *and* $\mathcal{D}_h$ *respectively. It is straight forward to check that* $\mathcal{D}_g = \langle M_3, M_4 \rangle^{\text{def}}$ *and* $\mathcal{D}_h = \langle M_2, M_3, M_4 \rangle^{\text{def}}$. *Furthermore,* $g^\vee \cong \phi_{13} : M_1 \to M_3$ *and* $h^\vee \cong \phi_{14} : M_1 \to M_4$ *and one can check that* $\mathcal{D}_g^\vee = \langle M_1, M_2 \rangle^{\text{def}}$ *and* $\mathcal{D}_h^\vee = \langle M_1, M_2, M_3 \rangle^{\text{def}}$. *Therefore we have* $\langle M_2 \rangle^{\text{def}} = \mathcal{D}_h \cap \mathcal{D}_g^\vee$ *and* $(\langle M_2 \rangle^{\text{def}})^\vee = \mathcal{D}_h^\vee \cap \mathcal{D}_g = \langle M_3 \rangle^{\text{def}}$. *We have shown that for any definable subcategory* $\mathcal{D} \subseteq kG\text{-}\underline{\text{Mod}}$ *and for* $i = 1, ..., 4$, *if* $M_i \in \mathcal{D}$ *then* $M_{5-i} \in \mathcal{D}^\vee$.

Example 7.3.2. *Suppose* $G = \langle g \mid g^5 = 1 \rangle$ *is the cyclic group of order five, let* k *be a field of characteristic* 5 *and let* M_i *for* $i = 1, ..., 5$ *denote the indecomposable pure-injectives, as in the above example. We show that the internal tensor-duality on the definable subcategories of* $kG\text{-}\underline{\text{Mod}}$ *and elementary duality of definable subcategories of* $kG\text{-Mod}$ *do not coincide. Given the pp pair* ϕ/ψ *where* $\psi(x)$ *is* $T^i x = 0$ *and* $\phi(x)$ *is* $T^{i+1}x = 0$, *the corresponding definable subcategory is generated by* $M_1, ..., M_i$, *as for all these,* T^i *annihilates the whole module. The elementary dual pp-pair is* $D\psi/D\phi$ *where* $D\phi(x)$: $\exists y, x = yT^{i+1}$ *and* $D\psi(x)$: $\exists y, x = yT^i$. *Therefore the elementary dual definable subcategory is generated by* $M_1, ..., M_i$. *That is, the definable subcategory generated by* $\{M_1, ..., M_i\}$ *for any* $i = 1, ..., 4$ *is self-dual with respect to elementary duality, whereas the internal tensor-dual is given by* $\{M_{5-i}, ..., M_4\}$ *which only coincides with* $\{M_1, ..., M_i\}$ *when* $i = 4$.

In rest of this section we will record some results from [4] which describe internal tensor-duality in the derived category of modules over a commutative ring.

Suppose R is a ring. We begin by considering two dualities.

Definition 7.3.3. [4, Section 2.2] Denote by $(-)^*$ the functor

$$\mathbf{R}\text{Hom}(-, R) : D(\text{Mod-}R) \to D(R\text{-Mod}).$$

As described in [4, Section 2.2], $(-)^*$ restricts to a duality between compact objects and when R is commutative $(-)^*$ is the internal hom-functor and on compact objects $(-)^* = (-)^\vee$, where $(-)^\vee$ is as in Definition 2.3.1.

Now assume R is a k-algebra for some commutative ring k and let W be an injective cogenerator in Mod-k (e.g. $k = \mathbb{Z}$ and $W = \mathbb{Q}/\mathbb{Z}$). Denote by $(-)^+$ the functor

$$\mathbf{R}\mathrm{Hom}_k(-, W) : D(\text{Mod-}R) \to D(R\text{-Mod}).$$

We will use the same notation for the quasi-inverse (contravariant) functors $(-)^* = \mathbf{R}\mathrm{Hom}(-, R) : D(R\text{-Mod}) \to D(\text{Mod-}R)$ and $(-)^+ = \mathbf{R}\mathrm{Hom}_k(-, W) : D(R\text{-Mod}) \to D(\text{Mod-}R)$.

Definition 7.3.4. (see [4, Lemma 2.3]) Given a definable subcategory $\mathcal{D} \subseteq D(\text{Mod-}R)$ with corresponding cohomological ideal $\mathcal{J} \subseteq \mathrm{morph}(D^c(\text{Mod-}R))$ we denote by $\mathcal{D}^* \subseteq D(R\text{-Mod})$ the definable subcategory corresponding to the cohomological ideal $\mathcal{J}^* \subseteq \mathrm{morph}(D^c(R\text{-Mod}))$.

Remarks 7.3.5. (i) The restriction of $(-)^*$ to compact objects is the duality D used in [27, Corollary 7.5] to give an inclusion-preserving bijective correspondence between the Serre subcategories of $D^c(\text{Mod-}R)$ and $D^c(R\text{-Mod})$.

(ii) $\mathcal{D}^*$ and $\mathcal{D}^\vee$ coincide in the case that R is commutative. In particular, the duality given in [27, Corollary 7.5] and the internal tensor-duality of Definition 7.1.3 coincide when R is a commutative ring and $\mathcal{T} = D(\text{Mod-}R)$.

The following lemma from [4] gives us a better understanding of internal tensor-duality in the case $\mathcal{T} = D(R\text{-Mod})$ for R a commutative ring.

Lemma 7.3.6. *[4, Lemma 2.3] Suppose $\mathcal{D}$ is a definable subcategory of $D(\text{Mod-}R)$.*

For every $X \in D(\text{Mod-}R)$, $X \in \mathcal{D}$ if and only if $X^+ \in \mathcal{D}^*$.

For every $Y \in D(R\text{-Mod})$, $Y \in \mathcal{D}^*$ if and only if $Y^+ \in \mathcal{D}$.

Chapter 8

Torsion pairs and definability

8.1 Internal tensor-duality of torsion pairs with definable coaisles

Recall that if $\mathcal{T} = D(R\text{-Mod})$ where R is a commutative ring, then internal tensor-duality coincides with the duality defined in [4, Lemma 2.3] (see Chapter 7). In this section we extend some results from [4] to the setting of algebraic rigidly-compactly generated tensor triangulated categories.

Recall (Definition 5.2.3) that a torsion pair $(\mathcal{U}, \mathcal{V})$ is a pair of full additive subcategories of $\mathcal{T}$ which are closed under direct summands, there are no morphisms from an object in $\mathcal{U}$ to an object in $\mathcal{V}$ and for every $X \in \mathcal{T}$, there exists an exact triangle $U \to X \to V \to \Sigma U$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. As a consequence both $\mathcal{U}$ and $\mathcal{V}$ are closed under extensions and therefore triangulated if and only if they are shift-closed. First let us consider stable and compactly generated torsion pairs with definable coaisles.

Definition 8.1.1. We say that a torsion pair $(\mathcal{U}, \mathcal{V})$ is **stable** if it is both a t-structure and a co-t-structure. In particular, both $\mathcal{U}$ and $\mathcal{V}$ are triangulated subcategories of $\mathcal{T}$.

The correspondence between triangulated definable subcategories and smashing subcategories in Theorem 5.2.10 can be rephrased as follows.

Theorem 8.1.2. *Suppose $(\mathcal{U}, \mathcal{V})$ is a stable torsion pair. Then $\mathcal{V}$ is definable if and only if $\mathcal{U}$ is a smashing subcategory of $\mathcal{T}$.*

Definition 8.1.3. A torsion pair $(\mathcal{U}, \mathcal{V})$ is said to be **generated** by a set of objects $\mathcal{X} \subseteq \mathcal{T}$ if $(\mathcal{U}, \mathcal{V}) = ({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$. If $\mathcal{X}$ is a set of compact objects, $(\mathcal{U}, \mathcal{V})$ is said to be **compactly generated**.

Remarks 8.1.4.
(i) Every compactly generated torsion pair has a definable coaisle since $\mathcal{V} = \{X \in \mathcal{T} : (\mathrm{id}_A, X) = 0 \ \forall A \in \mathcal{X}\}$. The associated cohomological ideal is generated by identity morphisms.

(ii) The Telescope Conjecture holds for $\mathcal{T}$ if and only if every stable torsion pair whose coaisle is definable, is a compactly generated torsion pair. The tensor-Telescope Conjecture holds for $\mathcal{T}$ if and only if every stable torsion pair whose coaisle is a definable tensor-ideal, is a compactly generated torsion pair.

Lemma 8.1.5. *[2, Theorem 4.3] For every set of compact objects $\mathcal{X} \subseteq \mathcal{T}^c$, $({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$ forms a torsion pair.*

Proposition 8.1.6. *Internal tensor-duality yields a bijection*

$$\left\{\begin{smallmatrix}\text{Compactly generated} \\ \text{t-structures}\end{smallmatrix}\right\} \leftrightarrow \left\{\begin{smallmatrix}\text{Compactly generated} \\ \text{co-t-structures}\end{smallmatrix}\right\}.$$

Proof. $(\mathcal{U}, \mathcal{V})$ is a compactly generated t-structure generated by $\mathcal{X} \subseteq \mathcal{T}^c$ if and only if the cohomological ideal $\mathcal{J}$ associated to the negative-shift-closed definable subcategory $\mathcal{V}$ is generated by $\{\mathrm{id}_A : \forall A \in \mathcal{X}\}$. By Lemma 7.2.1, $\mathcal{J}$ is generated by $\{\mathrm{id}_A : \forall A \in \mathcal{X}\}$ if and only if the cohomological ideal $\mathcal{J}^{\vee}$ associated to the definable subcategory $\mathcal{V}^{\vee}$ is generated by $\{\mathrm{id}_{A^{\vee}} : \forall A^{\vee} \in \mathcal{X}^{\vee}\}$. By Proposition 7.2.4, $\mathcal{V}$ is negative shift-closed if and only if $\mathcal{V}^{\vee}$ is positive shift-closed.

Thus, $({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$ is a compactly generated t-structure generated by $\mathcal{X} \subseteq \mathcal{T}^c$ if and only if $({}^{\perp}((\mathcal{X}^{\vee})^{\perp}), (\mathcal{X}^{\vee})^{\perp})$ is a compactly generated co-t-structure. $\square$

For the rest of this section we assume that $\mathcal{T}$ is algebraic. This assumption allows us to extend Proposition 8.1.6 to TTF triples (defined below).

Definition 8.1.7. A compactly generated triangulated category $\mathcal{T}$ is said to be **algebraic** if it is equivalent to the derived category of a small dg category or equivalently the stable category of a Frobenius exact category which is compactly generated (see for example [40, Section 3] for more details).

Definition 8.1.8. [4, Section 2.5] A **TTF (torsion-torsion-free) triple** $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is formed by two adjacent torsion pairs $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$. A TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be **suspended** (respectively **cosuspended**) if $\Sigma\mathcal{V} \subseteq \mathcal{V}$ (respectively $\Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$). A TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be **generated** by a set of objects $\mathcal{X}$ of $\mathcal{T}$ if $\mathcal{V} = \mathcal{X}^{\perp}$. If $\mathcal{X}$ is a set of compact objects, $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be **compactly generated**.

Suppose $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a TTF triple with $\mathcal{V}$ definable. Let $\mathcal{V}^{\vee}$ be the internal tensor-dual of $\mathcal{V}$ and set $\mathcal{U}' = {}^{\perp}\mathcal{V}^{\vee}$ and $\mathcal{W}' = (\mathcal{V}^{\vee})^{\perp}$. We consider whether $(\mathcal{U}', \mathcal{V}^{\vee}, \mathcal{W}')$ forms a TTF triple. We introduce the following terminology.

Definition 8.1.9. We will say that a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is **definable** if $\mathcal{V}$ is a definable subcategory of $\mathcal{T}$.

Under the assumption that $\mathcal{T}$ is an algebraic rigidly-compactly generated tensor triangulated category, we have the following lemma.

Lemma 8.1.10. *([2, Theorem 4.3] and [58, Theorem 3.11]) For every set $\mathcal{X}$ of compact objects, $({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp}, (\mathcal{X}^{\perp})^{\perp})$ is a TTF triple.*

Proposition 8.1.11. *Suppose $\mathcal{T}$ is an algebraic rigidly-compactly generated tensor triangulated category. Then internal tensor-duality of definable subcategories induces a bijective correspondence*

$$\left\{\begin{matrix}\text{Suspended compactly generated}\\ \text{TTF triples}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Cosuspended compactly generated}\\ \text{TTF triples}\end{matrix}\right\}.$$

Proof. By Lemma 8.1.10, for every set $\mathcal{X}$ of compact objects $({}^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp}, (\mathcal{X}^{\perp})^{\perp})$ is a TTF triple. Thus every compactly generated t-structure $(\mathcal{U}, \mathcal{V})$ extends to a cosuspended TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ and every compactly generated co-t-structure $(\mathcal{U}', \mathcal{V}')$ extends to a suspended TTF triple $(\mathcal{U}', \mathcal{V}', \mathcal{W}')$. The result then follows from Proposition 8.1.6. $\square$

Remarks 8.1.12. (i) For $\mathcal{T} = D(R\text{-Mod})$ where R is a commutative ring, this correspondence coincides with the 1-1 correspondence given in [4, Theorem 3.1] (see Theorem 8.2.14).

(ii) If $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ and $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$ correspond as in Proposition 8.1.11, then $\mathcal{U} \cap \mathcal{T}^c)^\vee = \mathcal{U}' \cap \mathcal{T}^c$.

Next we extend Proposition 8.1.11 to definable TTF triples. The following proposition will be useful.

Proposition 8.1.13. *[37, Lemma 4.2 and Theorem 12.1] Suppose a torsion pair $(\mathcal{U}, \mathcal{V})$ has a definable coaisle. Then the cohomological ideal corresponding to $\mathcal{V}$ is given by $\mathcal{J} = \{f \in \mathrm{morph}(\mathcal{T}^c) : f \text{ factors via some } U \in \mathcal{U}\}$.*

Proof. Clearly if $f \in \mathrm{morph}(\mathcal{T}^c)$ factors through some object in $\mathcal{U}$ then $f \in \mathcal{J}$. Conversely, suppose $f : A \to B$ is a morphism in $\mathcal{J}$. Since $(\mathcal{U}, \mathcal{V})$ is a torsion pair, we have an exact triangle $U \xrightarrow{g} B \xrightarrow{k} V \to \Sigma U$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $(f, V) = 0$ we must have that $k \circ f = 0$. But then since g is a weak kernel of k, f factors via g, say $f = g \circ f'$ as required. □

Lemma 8.1.14. *Let $\mathcal{D} \subseteq \mathcal{T}$ be a definable subcategory. If $\mathcal{D}$ is extension-closed then $({}^\perp\mathcal{D}, \mathcal{D})$ is a torsion pair. If, in addition, we assume that $\mathcal{T}$ is an algebraic triangulated category, then $(\mathcal{D}, \mathcal{D}^\perp)$ is also a torsion pair and $\mathcal{D}$ fits into a TTF triple $({}^\perp\mathcal{D}, \mathcal{D}, \mathcal{D}^\perp)$.*

Proof. It is well known that any definable subcategory $\mathcal{D} \subseteq \mathcal{T}$ is preenveloping [5, Proposition 4.5]. If in addition we assume that $\mathcal{T}$ is an algebraic triangulated category, then any definable subcategory $\mathcal{D} \subseteq \mathcal{T}$ is precovering [40, Corollary 4.8]. It remains to apply Corollary 5.2.7. □

Next we use results from [37] to prove the following lemma.

Lemma 8.1.15. *$\mathcal{D} \subseteq \mathcal{T}$ is extension-closed if and only if $\mathcal{D}^\vee \subseteq \mathcal{T}$ is extension-closed.*

Proof. First we show that a definable subcategory $\mathcal{D}$ is extension-closed if and only if the corresponding cohomological ideal $\mathcal{J} \subseteq \mathrm{morph}(\mathcal{T}^c)$ is idempotent.

Suppose $\mathcal{J} = \mathcal{J}^2$ and let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$ with $X, Z \in \mathcal{D}$. Assume $f : A \to B$ is an morphism in $\mathcal{J}$. Then $f = h \circ g$ for some $g : A \to A'$ and $h : A' \to B$ in $\mathcal{J}$. Therefore, given any morphism $k : B \to Y$, $\beta \circ k \circ h = 0$ as $Z \in \mathcal{D}$ meaning $k \circ h = \alpha \circ h'$ for some $h' : A' \to X$, as shown below.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[dd, "k"] & & \\
& A' \arrow[ur, "h"'] \arrow[dl, dashed, "h'"'] & & & \\
X \arrow[rr, "\alpha"] & & Y \arrow[r, "\beta"] & Z \arrow[r] & \Sigma X
\end{tikzcd}$$

Consequently, $k \circ f = k \circ h \circ g = \alpha \circ h' \circ g$, but $h' \circ g = 0$ as $X \in \mathcal{D}$, so $k \circ f = 0$ and $Y \in \mathcal{D}$. Therefore $\mathcal{D}$ is extension-closed.

Conversely, assume $\mathcal{D}$ is extension-closed and let $\mathcal{J}$ be the corresponding cohomological ideal. By Lemma 8.1.13,

$$\mathcal{J} = \{f \in \operatorname{morph}(\mathcal{T}^c) : f \text{ factors through some } X \in {}^{\perp}\mathcal{D}\}.$$

Following the proof of [37, Theorem 12.1], we show that any morphism $f : A \to U$ where $U \in {}^{\perp}\mathcal{D}$ factors through some $g \in \mathcal{J}$. Denote by $\mathsf{C} \subseteq \text{mod-}\mathcal{T}^c$ the Serre subcategory corresponding to $\mathcal{D}$, that is $\mathsf{C} = \{\text{im } H_f : f \in \mathcal{J}\}$, and set $\mathcal{L} = \varinjlim \mathsf{C}$. Then $\mathcal{L} \subseteq \text{Mod-}\mathcal{T}^c$ is a localising subcategory and $\text{Mod-}\mathcal{T}^c/\mathcal{L}$ is an abelian Grothendieck category. Let I be an injective cogenerator of $\text{Mod-}\mathcal{T}^c/\mathcal{L}$, let $q : \text{Mod-}\mathcal{T}^c \to \text{Mod-}\mathcal{T}^c/\mathcal{L}$ be the quotient functor and denote the right adjoint to q by $r : \text{Mod-}\mathcal{T}^c/\mathcal{L} \to \text{Mod-}\mathcal{T}^c$. Notice that $(H_{(-)}, r(I)) : \mathcal{T}^{\text{op}} \to \mathbf{Ab}$ is a homological functor which takes coproducts to products. Therefore by Brown's representability theorem [42, Theorem 3.1], there exists some $X \in \mathcal{T}$ such that $\text{Mod-}\mathcal{T}^c(H_X, r(I)) \cong \mathcal{T}(-, X)$. For all $g \in \mathcal{J}$, $(H_g, r(I)) = 0$, as $\text{im } H_g \in \mathsf{C} \subseteq \mathcal{L}$, so for all $g \in \mathcal{J}$, $(g, X) = 0$ meaning $X \in \mathcal{D}$. Therefore if $U \in {}^{\perp}\mathcal{D}$ then $(H_U, r(I)) \cong (U, X) = 0$ and consequently $\text{Mod-}\mathcal{T}^c/\mathcal{L}(H_U, I) = 0$ meaning $H_U \in$

$\mathcal{L}$. Therefore we can write H_U as a direct limit $H_U = \varinjlim \mathrm{im}\, H_{g_i}$ where each $g_i \in \mathcal{J}$. Suppose $f : A \to U$. Then there exists some $i \in I$ such that $H_f : H_A \to H_U$ factors through the colimit map $\lambda_i : \mathrm{im}\, H_{g_i} \to H_U$, say $H_f = \lambda_i \circ \gamma$. We have the following diagram.

$$\begin{tikzcd}
H_A \arrow[rr, "H_f"] \arrow[dr, "\gamma"] & & H_U \\
& \mathrm{im}\, H_{g_i} \arrow[ur, "\lambda_i"] \arrow[dr, "\iota"] & \\
H_{U_i} \arrow[ur] \arrow[rr, "H_{g_i}"] & & H_{V_i} \arrow[uu, dashed]
\end{tikzcd}$$

Recall that H_U is absolutely pure or equivalently fp-injective (Theorem 2.5.13), so λ_i factors through ι as shown by the dashed line in the above diagram. In addition, H_A is projective since $A \in \mathcal{T}^c$, so γ factors via H_{U_i} as pictured below.

$$\begin{tikzcd}
H_A \arrow[rr, "H_f"] \arrow[dr, "\gamma"] \arrow[dd, dashed] & & H_U \\
& \mathrm{im}\, H_{g_i} \arrow[ur, "\lambda_i"] \arrow[dr, "\iota"] & \\
H_{U_i} \arrow[ur] \arrow[rr, "H_{g_i}"] & & H_{V_i} \arrow[uu, dashed]
\end{tikzcd}$$

By Yoneda's lemma, the dashed lines on the above diagram are of the form $(-, k)$ and $(-, l)$ where $k : A \to U_i$ and $l : V_i \to U$. Therefore $f = l \circ g_i \circ k$ and in particular f factors via some $g_i : U_i \to V_i$ in $\mathcal{J}$.

We have shown that any morphism in $\mathcal{J}$ factors as $A \xrightarrow{f} U \xrightarrow{f'} B$ for some $U \in {}^{\perp}\mathcal{D}$ and $f : A \to U$ factors as $A \xrightarrow{k} U_i \xrightarrow{g_i} V_i \xrightarrow{l} U$ for some $g_i \in \mathcal{J}$. Since $A \in \mathcal{T}^c$ and $\mathcal{J}$ is an ideal $g_i \circ k \in \mathcal{J}$ and since $f' \circ l : V_i \to B$ is a morphism in $\mathcal{T}^c$ which factors via $U \in {}^{\perp}\mathcal{D}$, $f' \circ l \in \mathcal{J}$. Therefore $\mathcal{J} = \mathcal{J}^2$ as required.

It remains to note that $\mathcal{J} = \mathcal{J}^2$ if and only if $\mathcal{J}^{\vee} = (\mathcal{J}^{\vee})^2$. $\square$

Theorem 8.1.16. *Let $\mathcal{T}$ be an algebraic rigidly-compactly generated tensor triangulated category. Then internal tensor-duality induces a bijection*

$$\left\{\begin{matrix}\text{Suspended definable}\\ \text{TTF triples in } \mathcal{T}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Cosuspended definable}\\ \text{TTF triples in } \mathcal{T}\end{matrix}\right\}$$

which restricts to an automorphism on the class

$$\left\{\begin{matrix}\text{Stable definable}\\ \text{TTF triples in } \mathcal{T}\end{matrix}\right\}$$

and restricts to a bijection

$$\left\{\begin{matrix}\text{Suspended compactly generated}\\ \text{TTF triples in } \mathcal{T}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Cosuspended compactly generated}\\ \text{TTF triples in } \mathcal{T}\end{matrix}\right\}.$$

Proof. Suppose $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a suspended definable TTF triple. Then $\Sigma\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V}$ is extension-closed. By Lemma 8.1.15, $\mathcal{V}^\vee$ is also extension-closed and by Proposition 7.2.4 $\mathcal{V}^\vee$ is negative shift-closed. Thus $\mathcal{V}^\vee$ is cosuspended. Applying Lemma 8.1.14 we get a cosuspended definable TTF triple $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$. Conversely, if $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a cosuspended definable TTF triple, then $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$ is a suspended definable TTF triple by a similar argument. A TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is stable and definable if and only if $\mathcal{V}$ is a shift-closed definable subcategory. In this case, by Proposition 7.2.4, $\mathcal{V}^\vee$ is a shift-closed definable subcategory and by Lemma 8.1.14, $\mathcal{V}^\vee$ can be extended to a TTF triple $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$. The restriction to compactly generated TTF triples is Proposition 8.1.11. $\square$

8.2 Silting and cosilting objects

In this section we consider the case $\mathcal{T} = D(R\text{-Mod})$ where R is a commutative ring. We describe how the bijection in Theorem 8.1.16 restricts to an injective map from certain silting objects to pure-injective cosilting objects. In turn this injective map restricts to the silting-cosilting duality established in [4, Theorem 3.1 and Theorem 3.3].

Notation 8.2.1. Recall from Notation 5.2.1 that for any $I \subseteq \mathbb{Z}$, $\mathcal{X}^{\perp_I} = \{Z \in \mathcal{T} :$

$(X, \Sigma^i Z) = 0, \forall X \in \mathcal{X}, i \in I\}$ and ${}^{\perp_I}\mathcal{X} = \{Z \in \mathcal{T} : (Z, \Sigma^i X) = 0, \forall X \in \mathcal{X}, i \in I\}$. If $I = \{i \in \mathbb{Z} : i > 0\}$ we will write $\perp > 0$ and if $I = \{i \in \mathbb{Z} : i \leq 0\}$ we will write $\perp \leq 0$.

Definition 8.2.2. [52, Definition 4.1] An object $S \in \mathcal{T}$ is **silting** if $(S^{\perp > 0}, S^{\perp \leq 0})$ is a t-structure in $\mathcal{T}$. An object $C \in \mathcal{T}$ is **cosilting** if $({}^{\perp \leq 0}C, {}^{\perp > 0}C)$ is a t-structure in $\mathcal{T}$. Two silting (respectively cosilting) objects S and S' in a triangulated category with coproducts are said to be **equivalent** if they induce the same t-structure.

Example 8.2.3. *[52, Example 4.2] Let $\mathcal{A}$ be an abelian category with a projective generator P. Then P is a silting (in fact tilting) object in $D(\mathcal{A})$ and the associated t-structure is the standard one. Dually, if $\mathcal{A}$ has an injective cogenerator E then E is cosilting (in fact cotilting) in $D(\mathcal{A})$ and the associated cotilting t-structure is also the standard one.*

Definition 8.2.4. [4, Section 2.5] A torsion pair $(\mathcal{U}, \mathcal{V})$ is said to be **non-degenerate** if $\bigcap_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = 0 = \bigcap_{n \in \mathbb{Z}} \Sigma^n \mathcal{V}$. A suspended TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be **non-degenerate** if so is the t-structure $(\mathcal{V}, \mathcal{W})$. A cosuspended TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be **non-degenerate** if so is the t-structure $(\mathcal{U}, \mathcal{V})$.

Definition 8.2.5. A subset $\mathcal{X} \subseteq D(R\text{-Mod})$ is said to be **closed under directed homotopy colimits** if for every directed diagram of chain complexes, $\{X_i : i \in I\}$, in $\mathrm{Ch}(R\text{-Mod})$ such that (when viewed as an element of $D(R\text{-Mod})$), each X_i is in $\mathcal{X}$, then the direct limit $\varinjlim_{i \in I} X_i$ (calculated in $\mathrm{Ch}(R\text{-Mod})$) belongs to $\mathcal{X}$.

A TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be **homotopically smashing** if $\mathcal{V}$ is closed under directed homotopy colimits.

The following result characterises silting and cosilting t-structures in terms of non-degenerate TTF triples.

Proposition 8.2.6. *[3, Theorem 4.11 and Theorem 6.13] There is a bijective correspondence between silting t-structures $(\mathcal{V}, \mathcal{W})$ and non-degenerate suspended TTF triples $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ which are generated by a set of objects.*

If $\mathcal{T}$ is an algebraic compactly-generated triangulated category, then there is a bijective correspondence between the t-structures $(\mathcal{U}, \mathcal{V})$ which are generated by a *pure-injective cosilting object and non-degenerate cosuspended TTF triples $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ which are homotopically smashing.*

In addition we have the following.

Proposition 8.2.7. *[39, Theorem 4.6] Suppose $\mathcal{T}$ is an algebraic compactly-generated triangulated category and $t = (\mathcal{U}, \mathcal{V})$ is a non-degenerate t-structure. Then t is generated by a pure-injective cosilting object if and only if $\mathcal{V}$ is definable.*

Lemma 8.2.8. *Suppose $\mathcal{V} \subseteq \mathcal{T}$ is definable and extension-closed. Then $\bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{V} = 0$ if and only if $\bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{V}^\vee = 0$.*

Proof. Suppose $\mathcal{V} \subseteq \mathcal{T}$ is definable and extension-closed and $\bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{V} = 0$. Then by Lemma 6.1.1, $\big\langle \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{J} \big\rangle^{\text{cohom}} = \text{morph}(\mathcal{T}^c)$ where $\mathcal{J}$ is the cohomological ideal associated to $\mathcal{V}$. But then $\big\langle \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{J}^\vee \big\rangle^{\text{cohom}} = \big\langle (\bigcup_{n\in\mathbb{Z}} \Sigma^{-n} \mathcal{J})^\vee \big\rangle^{\text{cohom}} = (\big\langle \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{J} \big\rangle^{\text{cohom}})^\vee = \text{morph}(\mathcal{T}^c)$, using Lemma 7.2.1. So $\bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{V}^\vee = 0$ as required. □

For the rest of this section we restrict to the case $\mathcal{T} = D(R\text{-Mod})$ where R is a commutative ring. Recall that $(-)^+$ denotes the functor $\mathbf{R}\text{Hom}_k(-, W) : D(\text{Mod-}R) \to D(R\text{-Mod}) \cong D(\text{Mod-}R)$ where R is a k-algebra for some commutative ring k and W is an injective cogenerator for Mod-k (Definition 7.3.3).

In this case non-degeneracy is preserved by one direction of the bijective correspondence given in Theorem 8.1.16.

Lemma 8.2.9. *[4, Lemma 3.2] Suppose $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a suspended TTF triple such that $\mathcal{V}$ is definable and let $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$ denote the dual cosuspended TTF triple with respect to Theorem 8.1.16. If $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ is non-degenerate then so is $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$.*

Proof. Suppose $X \in \bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{U}' = {}^{\perp_\mathbb{Z}}\mathcal{V}^\vee$. Then $\mathbf{R}\text{Hom}(X, \mathcal{V}^\vee) = 0$ since

$$H^n(\mathbf{R}\text{Hom}(X, \mathcal{V}^\vee)) \cong (\Sigma^n R, \mathbf{R}\text{Hom}(X, \mathcal{V}^\vee)) \cong (X, \Sigma^{-n} \mathcal{V}^\vee) = 0$$

for all $n \in \mathbb{Z}$. By Lemma 7.3.6 $\mathcal{V}^+ \subseteq \mathcal{V}^\vee$ so $\mathbf{R}\mathrm{Hom}(X, \mathcal{V}^+) = 0$. But by [4, Lemma 2.1], $\mathbf{R}\mathrm{Hom}(X, \mathcal{V}^+) \cong \mathbf{R}\mathrm{Hom}(\mathcal{V}, X^+)$ so $\mathbf{R}\mathrm{Hom}(\mathcal{V}, X^+) = 0$ meaning

$$0 = H^n(\mathbf{R}\mathrm{Hom}(\mathcal{V}, X^+)) \cong (\Sigma^n R, \mathbf{R}\mathrm{Hom}(\mathcal{V}, X^+)) \cong (\mathcal{V}, \Sigma^{-n} X^+)$$

for all $n \in \mathbb{Z}$. Therefore $X^+ \in \mathcal{V}^{\perp_\mathbb{Z}} = \bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{W} = 0$, i.e. $X^+ = 0$. Thus $X = 0$ and $\bigcap_{n\in\mathbb{Z}} \Sigma^n \mathcal{U}' = 0$ as required. □

Proposition 8.2.10. *Let $\mathcal{T} = D(R\text{-Mod})$ where R is a commutative ring. Internal tensor-duality gives rise to an injective map*

$$\left\{ \begin{array}{c} \text{Silting objects } S \text{ in } D(R\text{-Mod}) \\ \text{with } S^{\perp>0} \text{ definable, up to equivalence} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{Pure}-\text{injective cosilting objects} \\ \text{in } D(R\text{-Mod}), \text{ up to equivalence} \end{array} \right\}.$$

Proof. We have seen that every silting object gives rise to a non-degenerate suspended TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ by (Proposition 8.2.6). Therefore, if in addition $\mathcal{V} = S^{\perp>0}$ is definable we can apply Theorem 8.1.16 to get a cosuspended TTF triple $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$. By Lemma 8.2.8 and Lemma 8.2.9 we have that $(\mathcal{U}', \mathcal{V}^\vee, \mathcal{W}')$ is non-degenerate and therefore applying Proposition 8.2.6 again, we get a pure-injective cosilting object. □

Remark 8.2.11. The map in Proposition 8.2.10 is surjective if and only if the converse to Lemma 8.2.9 holds.

Next we give a result from [4] which, in the case that R is a commutative ring, uses a restriction of the injective map given in Proposition 8.2.10. First we need some definitions.

Definition 8.2.12. [4, Definition 2.13] A silting object $S \in D(\text{Mod-}R)$ is of **finite type** if the TTF triple it induces is compactly generated. Similarly, a cosilting object $C \in D(R\text{-Mod})$ is said to be of **cofinite type** if it induces a compactly generated TTF triple.

Definition 8.2.13. [4, Section 2.6] A silting object in $D(\text{Mod-}R)$ is called a **bounded silting complex** if it belongs to $K^b(\text{Proj-}R)$. Similarly, a cosilting object in $D(R\text{-Mod})$ is a **bounded cosilting complex** if it belongs to $K^b(R\text{-Inj})$.

The following theorem from [4] shows how the correspondence in Proposition 8.2.10 restricts to a silting-cosilting duality.

Theorem 8.2.14. *[4, Theorem 3.1 and Theorem 3.3] Suppose R is any ring. There is a one-to-one correspondence*

$$\left\{\begin{matrix}\text{Compactly generated}\\ \text{TTF}-\text{triples in } D(\text{Mod-}R)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Compactly generated}\\ \text{TTF}-\text{triples in } D(R\text{-Mod})\end{matrix}\right\},$$

given by mapping the TTF triple in $D(\text{Mod-}R)$ generated by the set S of compact objects to the TTF triple in $D(R\text{-Mod})$ generated by $S^* = \{C^* : C \in S\}$.

Furthermore, this correspondence induces an injective map

$$\left\{\begin{matrix}\text{Silting objects of finite type}\\ \text{in } D(\text{Mod-}R)\text{, up to equivalence}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{Cosilting objects of cofinite type}\\ \text{in } D(R\text{-Mod})\text{, up to equivalence}\end{matrix}\right\},$$

which is given by $S \mapsto S^+$ and restricts to a bijection

$$\left\{\begin{matrix}\text{Bounded silting complexes}\\ \text{in } D(\text{Mod-}R)\text{, up to equivalence}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Bounded cosilting complexes of cofinite type}\\ \text{in } D(R\text{-Mod})\text{, up to equivalence}\end{matrix}\right\}.$$

8.3 T-structures with monoidal hearts

In this section we take a closer look at the example $\mathcal{T} = D(R\text{-Mod})$ where R is a coherent commutative ring of weak global dimension at most one such that every finitely presented R-module has finite projective dimension. Here $\mathcal{T}$ comes equipped with the standard t-structure which has the monoidal heart R-Mod. We show that for each definable subcategory $\mathcal{D}$ of $\mathcal{T}$ and each $n \in \mathbb{Z}$, $H^n(\mathcal{D})$ and $H^{-n}(\mathcal{D}^\vee)$ are elementary dual definable subcategories of R-Mod.

We begin with some background on t-structures in compactly generated triangulated categories. Suppose $(\mathcal{U}, \mathcal{V})$ is a t-structure. Then $\mathcal{U}$ is suspended and precovering and $\mathcal{V}$ is cosuspended and preenveloping. Therefore we can apply Proposition 5.2.6, which results in the following adjoint functors.

Definition 8.3.1. We define the **truncation functors** $\tau_{\mathcal{U}} : \mathcal{T} \to \mathcal{U}$ (respectively $\tau_{\mathcal{V}} : \mathcal{T} \to \mathcal{V}$) to be the right adjoint to the inclusion functor $\mathcal{U} \hookrightarrow \mathcal{T}$ (respectively left adjoint to the inclusion functor $\mathcal{V} \hookrightarrow \mathcal{T}$).

It follows from the existence of these truncation functors that the triangle in part (iii) of the definition of a torsion pair is actually functorial and can be written as $\tau_{\mathcal{U}}(X) \to X \to \tau_{\mathcal{V}}(X) \to \Sigma\tau_{\mathcal{U}}(X)$.

Recall that the heart of the t-structure $(\mathcal{U}, \mathcal{V})$ is given by $\mathcal{H}_t = \mathcal{U} \cap \Sigma\mathcal{V}$. Given a t-structure $t = (\mathcal{U}, \mathcal{V})$ there exists a functor $H_t^0 : \mathcal{T} \to \mathcal{H}_t$ given by $H_t^0 = \tau_{\mathcal{U}} \circ \Sigma \circ \tau_{\mathcal{V}} \circ \Sigma^{-1}$. In addition, we set $H_t^n = H_t^0 \circ \Sigma^n$ for any integer n. We have the following proposition.

Proposition 8.3.2. *(e.g. see [32, Proposition 10.1.11(i)]) The heart of a t-structure, $t = (\mathcal{U}, \mathcal{V})$, is an abelian category and $H_t^0 : \mathcal{T} \to \mathcal{H}_t$ is a cohomological functor.*

Below we give a standard example which motivates these definitions.

Example 8.3.3. *Let $\mathcal{A}$ be an abelian category. The* ***standard t-structure*** *in $D(\mathcal{A})$ is given by $\mathbb{D} = (\mathbb{D}^{\leq 0}, \mathbb{D}^{\geq 1})$ where $\mathbb{D}^{\leq 0} = \{X \in D(\mathcal{A}) : H_0^i(X) = 0,\ \forall i > 0\}$ and $\mathbb{D}^{\geq 1} = \{X \in D(\mathcal{A}) : H_0^i(X) = 0,\ \forall i < 1\}$ where $H_0^i : D(\mathcal{A}) \to \mathcal{A}$ denotes the usual ith cohomology functor.*

The heart of the standard t-structure is $\mathbb{D}^{\leq 0} \cap \mathbb{D}^{\geq 0} = \{X \in D(\mathcal{A}) : H_0^i(X) = 0,\ \forall i \neq 0\}$ and is therefore equivalent to the category $\mathcal{A}$. Furthermore the cohomological functor $H_{\mathbb{D}}^n : D(\mathcal{A}) \to \mathcal{H}_{\mathbb{D}} \simeq \mathcal{A}$ is the nth cohomology functor.

In particular, if $\mathcal{A} = R\text{-Mod}$ for some commutative ring R then $D(R\text{-Mod})$ is a rigidly-compactly generated tensor triangulated category and the heart has an additive closed symmetric monoidal structure given by the tensor product of R-modules, $\otimes_R$.

Next we give some results comparing the Ziegler spectrum of $D(R\text{-Mod})$ with the Ziegler spectrum of R-Mod. We begin with the following definition using the notation of [27].

Definition 8.3.4. Define a functor $I_n : R\text{-Mod} \to D(R\text{-Mod})$ given on objects by $M \mapsto M[-n]$, for each $n \in \mathbb{Z}$.

Proposition 8.3.5. *([27, Proposition 7.1]) For each $n \in \mathbb{Z}$, the functor $I_n : R\text{-Mod} \to D(R\text{-Mod})$ maps indecomposable pure-injective R-modules to indecomposable pure-injective objects in $D(R\text{-Mod})$.*

Definition 8.3.6. We denote by $\mathrm{Zg}_R^n \subseteq \mathrm{Zg}_{D(R\text{-Mod})}$ the set of (isomorphism classes of) indecomposable pure-injectives in the image of the restriction of I_n to $\mathrm{Zg}(R\text{-Mod})$.

The following theorem was proven in [27].

Theorem 8.3.7. *([27, Theorem 7.3]) Let R be a ring. The following statements hold.*

(i) For each $n \in \mathbb{Z}$, Zg_R^n is a closed subset of $\mathrm{Zg}_{D(R\text{-Mod})}$.

(ii) Suppose R is right coherent and every finitely presented R-module has finite projective dimension. Then I_n induces a homeomorphism between $\mathrm{Zg}(R\text{-Mod})$ and $\mathrm{Zg}_R^n \subseteq \mathrm{Zg}_{D(R\text{-Mod})}$ with the subspace topology.

(iii) The disjoint union $\bigcup_{n\in\mathbb{Z}} \mathrm{Zg}_R^n$ is a closed subset in $\mathrm{Zg}_{D(R\text{-Mod})}$ with open complement $\mathcal{X}$ consisting of the indecomposable pure-injective complexes with at least two non-zero cohomology groups. Thus,

$$\mathrm{Zg}_{D(R\text{-Mod})} = \mathcal{X} \cup \bigcup_{n\in\mathbb{Z}} \mathrm{Zg}_R^n.$$

In [27] it is shown using the above theorem, that $\mathrm{Zg}_{D(R\text{-Mod})} = \bigcup_{n\in\mathbb{Z}} \mathrm{Zg}_R^n$ for R von Neumann regular or right hereditary. In [13] the authors give the following generalisation.

Theorem 8.3.8. *[13, Theorem 3.4 and Corollary 3.6] Let R be a ring of weak global dimension at most one. Then every definable subcategory $\mathcal{D} \subseteq D(R\text{-Mod})$ is determined on cohomology, that is $X \in \mathcal{D}$ if and only if $H^nX[-n] \in \mathcal{D}$ for all $n \in \mathbb{Z}$. As a result we have $\mathrm{Zg}_{D(R\text{-Mod})} = \bigcup_{n\in\mathbb{Z}} \mathrm{Zg}_R^n$.*

Throughout the rest of this subsection suppose R is a coherent commutative ring of weak global dimension at most one such that every finitely presented R-module has finite projective dimension and set $\mathcal{T} = D(R\text{-Mod})$. Then for every $M \in R\text{-mod}$ and $n \in \mathbb{Z}$, $M[-n] \in D^c(R\text{-Mod})$. By Theorem 8.3.7, every I_n

induces a homeomorphism between $\mathrm{Zg}(R\text{-Mod})$ and $\mathrm{Zg}^n_R \subseteq \mathrm{Zg}_{D(R\text{-Mod})}$ with the subspace topology and by Theorem 8.3.8, $\mathrm{Zg}_{D(R\text{-Mod})} = \bigcup_{n\in\mathbb{Z}} \mathrm{Zg}^n_R$.

Proposition 8.3.9. *Suppose $\mathcal{D} \subseteq D(R\text{-Mod})$ is a $\mathcal{T}$-tensor-closed definable subcategory, then $H^n(\mathcal{D}) \subseteq R\text{-Mod}$ is an fp-hom-closed definable subcategory.*

Proof. Suppose $X \in H^n(\mathcal{D})$ and $A \in R\text{-mod}$. By our assumption on R and Theorems 8.3.7 and 8.3.8, $H^n(\mathcal{D})$ is definable, $X[-n] \in \mathcal{D}$ and $A[0] \in D^c(R\text{-Mod})$. Since $\mathcal{D}$ is $\mathcal{T}$-tensor-closed, $A[0]^\vee \otimes^{\mathbf{L}}_R X[-n] \cong \mathbf{R}\mathrm{Hom}(A[0], X[-n]) \in \mathcal{D}$ and therefore $H^n(\mathbf{R}\mathrm{Hom}(A[0], X[-n])) \in H^n(\mathcal{D})$. But,

$$H^n(\mathbf{R}\mathrm{Hom}(A[0], X[-n])) \cong D(R\text{-Mod})(A[0], X[0]) \cong \mathrm{Hom}_R(A, X),$$

so $\mathrm{hom}(A, X) \in H^n(\mathcal{D})$ and $H^n(\mathcal{D})$ is fp-hom-closed. $\square$

Corollary 8.3.10. *There exists a lattice monomorphism*

$$\mathbb{O}(\mathrm{Zg}^{\otimes}_{D(R\text{-Mod})}) \hookrightarrow \mathbb{O}(\bigcup_{n\in\mathbb{Z}} \mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})),$$

where $\bigcup_{n\in\mathbb{Z}} \mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})$ denotes the $\mathbb{Z}$-indexed disjoint union of copies of the fp-hom-closed Ziegler topology. The mapping on closed complements sends a closed subset $\mathscr{C} \subseteq \mathrm{Zg}^{\otimes}_{D(R\text{-Mod})}$ to $\bigcup_{n\in\mathbb{Z}} \mathscr{C}^n \subseteq \bigcup_{n\in\mathbb{Z}} \mathrm{Zg}^{\mathrm{hom}}(R\text{-Mod})$ where, if $\mathscr{C} = \mathcal{D} \cap \mathrm{pinj}_{\mathcal{T}}$ for a $\mathcal{T}$-tensor-closed definable subcategory $\mathcal{D} \subseteq \mathcal{T}$, then $\mathscr{C}^n = H^n(\mathcal{D}) \cap \mathrm{pinj}_{R\text{-Mod}}$.

Proposition 8.3.11. *If $\mathcal{D} \subseteq D(R\text{-Mod})$ is a definable subcategory with internal tensor-dual $\mathcal{D}^\vee$, then for any $n \in \mathbb{Z}$, $(\mathcal{D} \cap I_n(R\text{-Mod}))^\vee = \mathcal{D}^\vee \cap I_{-n}(R\text{-Mod})$.*

Proof. Suppose $\mathcal{D} \subseteq D(R\text{-Mod})$ is a definable subcategory with corresponding cohomological ideal $\mathcal{J}$. Then $\mathcal{D} \cap I_n(R\text{-Mod})$ is also definable by Theorem 8.3.7 [27, Theorem 7.3] and we will denote the corresponding cohomological ideal by $\mathcal{J}'$. Since, for all $X \in \mathcal{D} \cap I_n(R\text{-Mod})$, $H^i(X) = 0$ for all $i \neq n$, $(R[-i], X) = 0$ for all $i \neq n$. Therefore, $\mathrm{id}_{R[-i]} \in \mathcal{J}'$ for all $i \neq n$ and $\mathcal{J}'$ is generated as a cohomological ideal by $\mathcal{J} \cup \{R[i] : i \neq -n\}$. Therefore, by Lemma 7.2.1 and noting that $(\mathrm{id}_{R[-i]})^\vee = \mathrm{id}_{R[-i]^\vee} = \mathrm{id}_{R[i]}$, the cohomological ideal $\mathcal{J}'^\vee$ associated to $(\mathcal{D} \cap I_n(R\text{-Mod}))^\vee$ is generated by $\mathcal{J}^\vee \cup \{R[i] : i \neq n\}$.

Therefore, $Y \in (\mathcal{D} \cap I_n(R\text{-Mod}))^\vee$ if and only if $Y \in \mathcal{D}^\vee$ and $H^{-i}(Y) \cong (R[i], Y) \cong (\mathrm{id}_{R[i]}, Y) = 0$ for all $i \neq n$. So $Y \in (\mathcal{D} \cap I_n(R\text{-Mod}))^\vee$ if and only if $Y \in \mathcal{D}^\vee \cap I_{-n}(R\text{-Mod})$, as required. □

Corollary 8.3.12. *If $\mathcal{D} \subseteq I_n(R\text{-Mod}) \subseteq D(R\text{-Mod})$ is a definable subcategory then its internal tensor-dual $\mathcal{D}^\vee$ is contained in $I_{-n}(R\text{-Mod})$.*

Theorem 8.3.13. *Suppose R is a coherent commutative ring of weak global dimension at most one such that every finitely presented module is of finite projective dimension and let $\mathcal{T} = D(R\text{-Mod})$. Let $\mathcal{D}$ be a definable subcategory of $\mathcal{T}$ with internal tensor-dual $\mathcal{D}^\vee$. Then, for each $n \in \mathbb{Z}$, $H^n(\mathcal{D})$ and $H^{-n}(\mathcal{D}^\vee)$ are elementary dual definable subcategories in the sense of Theorem 2.4.12.*

Proof. Let $\mathcal{D}$ be a definable subcategory of $\mathcal{T}$ with corresponding cohomological ideal $\mathcal{J}$. First we show that $H^{-n}(\mathcal{D}^\vee) \subseteq H^n(\mathcal{D})^d$.

Let $Y \in H^{-n}(\mathcal{D}^\vee)$ and suppose $F_g \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ has the following presentation, $(V, -) \xrightarrow{(g,-)} (U, -) \to F_g \to 0$. Recall that $Y \in H^n(\mathcal{D})^d$ if and only if for all $F_g \in (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ such that $F_g(H^n(\mathcal{D})) = 0$, $(F_g)^d(Y) = 0$ where $(F_g)^d$ has presentation

$$0 \to (F_g)^d \to (U \otimes_R -) \xrightarrow{g \otimes_R -} (V \otimes_R -)$$

and $H^n(\mathcal{D})^d$ denotes the elementary dual definable subcategory of $H^n(\mathcal{D}) \subseteq R\text{-Mod}$. Assume $F_g(H^n(\mathcal{D})) = 0$, or equivalently for all $X \in H^n(\mathcal{D})$ and every morphism $h : U \to X$, h factors via g. We want to show that $g \otimes_R Y$ is a monomorphism.

By our assumption on R, $I_{-n}(g) : U[-n] \to V[-n]$ is a morphism in $\mathcal{T}^c$ and therefore there exists an exact triangle in $\mathcal{T}^c$ of the form

$$A \xrightarrow{f} U[-n] \xrightarrow{g[-n]} V[-n] \to A[1].$$

Now for every $Z \in \mathcal{D} \cap I_n(R\text{-Mod})$, $Z \cong H^n(Z)[-n]$ where $H^n(Z) \in H^n(\mathcal{D})$ and so any morphism $U[-n] \to Z$ corresponds (via I_n) to a morphism $U \to H^n(Z)$.

Therefore, since every morphism $U \to H^n(Z)$ factors via g, every morphism $U[-n] \to Z$ factors via $g[-n]$, or equivalently, $(f, Z) = 0$. Hence, f is in the cohomological ideal associated to $\mathcal{D} \cap I_n(R\text{-Mod})$, which we will denote by $\mathcal{J}'$.

Now, by [27, Theorem 7.3], $Y \in H^{-n}(\mathcal{D}^\vee)$ implies $Y[n] \in \mathcal{D}^\vee \cap I_{-n}(R\text{-Mod})$ and by Proposition 8.3.11 we have $Y \in H^{-n}(\mathcal{D}^\vee)$ if and only if $D(R\text{-Mod})(f^\vee, Y[n]) = 0$ for all $f \in \mathcal{J}'$. So $D(R\text{-Mod})(f^\vee, Y[n]) = 0$ for all $f \in \mathcal{J}'$ or equivalently $H^n(f \otimes^{\mathbf{L}}_R Y[0]) = 0$ for all $f \in \mathcal{J}'$.

Since $- \otimes^{\mathbf{L}}_R Y[0]$ is exact and H^n is cohomological we have an exact sequence

$$H^n(A \otimes^{\mathbf{L}}_R Y[0]) \xrightarrow{H^n(f \otimes^{\mathbf{L}}_R Y[0])} H^n(U[-n] \otimes^{\mathbf{L}}_R Y[0]) \xrightarrow{H^n(g[-n] \otimes^{\mathbf{L}}_R Y[0])} H^n(V[-n] \otimes^{\mathbf{L}}_R Y[0]).$$

Therefore since $f \in \mathcal{J}'$ and $H^n(f \otimes^{\mathbf{L}}_R Y[0]) = 0$, $H^n(g[-n] \otimes^{\mathbf{L}}_R Y[0])$ is a monomorphism. Consequently,

$$g \otimes_R Y \cong H^0(g[0] \otimes^{\mathbf{L}}_R Y[0]) \cong H^0((g[-n] \otimes^{\mathbf{L}}_R (Y[0]))[n]) \cong H^n((g[-n] \otimes^{\mathbf{L}}_R Y[0]))$$

is a monomorphism and $Y \in H^n(\mathcal{D})^d$.

We have established that $H^{-n}(\mathcal{D}^\vee) \subseteq H^n(\mathcal{D})^d$. For the converse, note that $H^n(\mathcal{D}) = H^n(\mathcal{D}^{\vee\vee}) \subseteq H^n(\mathcal{D}^\vee)^d$ and recall that elementary duality of definable subcategories is inclusion-preserving. Therefore, $H^n(\mathcal{D})^d \subseteq H^n(\mathcal{D}^\vee)$ and we have equality, as required. □

Remark 8.3.14. Under the assumptions of Theorem 8.3.13, if $\mathcal{D}$ is $\mathcal{T}$-tensor-closed, then $\mathcal{D}^\vee = \mathcal{D}$ (see Proposition 7.2.2) and we can see that the monomorphism in Corollary 8.3.10 is not an isomorphism. Indeed, given a non-zero fp-hom-closed definable subcategory $\mathcal{D} \subseteq R\text{-Mod}$ and $n \neq 0$, there is no $\mathcal{T}$-tensor-closed definable subcategory $\mathcal{X} \subseteq D(R\text{-Mod})$ such that $H^i(\mathcal{X}) = \begin{cases} 0 & \text{if } i \neq n \\ \mathcal{D} & \text{if } i = n, \end{cases}$ since if $H^n(\mathcal{X}) = \mathcal{D}$ then $H^{-n}(\mathcal{X}) = \mathcal{D}^d$.

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Appendix A

GAP code

A.1 Example 5.1.12

We show that $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a) \cong P \oplus P \oplus P \oplus M(ab^{-1})$, where $P \cong kV_4$ denotes the four dimensional indecomposable projective module. First we load the 'QPA' package (see [28]) and define the path algebra in GAP (see [26]) using the following input.

```
gap> LoadPackage("qpa");
gap> Q:=Quiver(1,[[1,1,"a"],[1,1,"b"]]);
gap> kQ:=PathAlgebra(Field(Z(2)),Q);
gap> AssignGeneratorVariables(kQ);
gap> relations:=[a^2,a*b-b*a,b^2];
gap> A:=kQ/relations;
```

Next we calculate by hand the action of a and b on the 15 dimensional module $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a)$. Denote the generators of $M(ab^{-1}ab^{-1})$ by $x_0, ..., x_4$ as pictured below.

$$\begin{tikzcd}
 & x_3 \arrow[dl, "a"'] \arrow[dr, "b"] & & x_1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\
x_4 & & x_2 & & x_0
\end{tikzcd}$$

Denote the generators of $M(b^{-1}a)$ by y_0, y_1 and y_2 with the action of a and b as pictured below.

$$\begin{tikzcd}
y_2 \arrow[dr, "b"] & & y_0 \arrow[dl, "a"'] \\
& y_1 &
\end{tikzcd}$$

Therefore we take $x_i \otimes y_j$ for $i = 0, ..., 4$ and $j = 0, ..., 2$ as generators for $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a)$. Fix the order of the generators to be

$$x_0 \otimes y_0,\ x_0 \otimes y_1,\ x_0 \otimes y_3,\ ...,\ x_4 \otimes y_0,\ x_4 \otimes y_1,\ x_4 \otimes y_2.$$

The action of a is given by multiplying on the left by the following matrix.

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The action of b is given by multiplying on the left by the following matrix.

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}$$

Below we provide the GAP input to define the matrices which correspond to the action of a and b, denoted by 'mata' and 'matb' respectively.

```
gap> v0:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v2:=[0*Z(2),Z(2)^0,0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v578:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0,0*Z(2),Z(2)^0,Z(2)^0,
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v8:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0,
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v9:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
Z(2)^0,0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v111314:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
0*Z(2),0*Z(2),0*Z(2),Z(2)^0,0*Z(2),Z(2)^0,Z(2)^0,0*Z(2)];
gap> v14:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
```

```
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0,0*Z(2)];
gap> v15:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0];
gap> mata:=[v2,v0,v0,v578,v8,v9,v8,v0,v0,v111314,v14,v15,v14,v0,v0];
gap> v1:=[Z(2)^0,0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v235:=[0*Z(2),Z(2)^0,Z(2)^0,0*Z(2),Z(2)^0,0*Z(2),0*Z(2),0*Z(2),
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v8:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0,
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v7:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),Z(2)^0,0*Z(2),
0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> v8911:=[0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),0*Z(2),
Z(2)^0,Z(2)^0,0*Z(2),Z(2)^0,0*Z(2),0*Z(2),0*Z(2),0*Z(2)];
gap> matb:=[v0,v0,v2,v1,v2,v235,v0,v0,v8,v7,v8,v8911,v0,v0,v14];
```

We define the 15 dimensional module $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a)$ assigned to the variable S as follows.

```
gap> S:=RightModuleOverPathAlgebra(A, [["a", mata],["b",matb]]);
<[ 15 ]>
```

Using the command ‘DecomposeModuleWithMultiplicities’ we find that $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a)$ decomposes into 3 copies of a 4-dimensional module and a 3-dimensional module.

```
gap> DecomposeModuleWithMultiplicities(S);
[ [ <[ 4 ]>, <[ 3 ]> ], [ 3, 1 ] ]
```

Recall that the unique indecomposable projective P is kG, where $G = V_4 = \langle x, y | x^2 = y^2 = [x, y] = e_G \rangle$. Here the generators are $1, x, y$ and xy, the action of $a \cong x + 1$ is given by multiplication on the left by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and the action of $b \cong y+1$ is given by multiplication on the left by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We define the 4-dimensional projective module in GAP, assigned to the variable P as follows.

```
gap> e12:=[Z(2)^0,Z(2)^0,0*Z(2),0*Z(2)];
gap> e34:=[0*Z(2),0*Z(2),Z(2)^0,Z(2)^0];
gap> e13:=[Z(2)^0,0*Z(2),Z(2)^0,0*Z(2)];
gap> e24:=[0*Z(2),Z(2)^0,0*Z(2),Z(2)^0];
gap> P:=RightModuleOverPathAlgebra(A,[["a",[e12,e12,e34,e34]],["b",
[e13,e24,e13,e24]]]);
```

Similarly the 3-dimensional module $M(ab^{-1})$ can be pictured as follows.

$$\begin{tikzcd} & x_1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\ x_2 & & x_0 \end{tikzcd}$$

Therefore, fixing the order of the generators x_0, x_1, x_2, the action of a is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the action of b is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we define $M(ab^{-1})$ in GAP, assigned to the variable H as follows.

```
gap> H:=RightModuleOverPathAlgebra(A, [["a", [[0*Z(2),0*Z(2),0*Z(2)],
[0*Z(2),0*Z(2),Z(2)^0],[0*Z(2),0*Z(2),
0*Z(2)]]],["b",[[0*Z(2),0*Z(2),0*Z(2)],[Z(2)^0,0*Z(2),0*Z(2)],
[0*Z(2),0*Z(2),0*Z(2)]]]]);
```

Now using the 'IsDirectSummand' command, we check that $P = kG$ and $H = M(ab^{-1})$ are indeed the 3 and 4-dimensional indecomposable direct summands of $M(ab^{-1}ab^{-1}) \otimes M(b^{-1}a)$.

```
gap> IsDirectSummand(P,S);
true

gap> IsDirectSummand(H,S);
true
```

A.2 Example 6.1.23

Let $\mathcal{T} = kG$-<u>Mod</u> where k is a field of characteristic 5 and $G = \langle g|g^5 = 1\rangle$. We show that the table below gives the tensor product over k of these modules.

$\otimes_k$	M_1	M_2	M_3	M_4	M_5
M_1	M_1	M_2	M_3	M_4	M_5
M_2	M_2	$M_1 \oplus M_3$	$M_2 \oplus M_4$	$M_3 \oplus M_5$	$M_5^{(2)}$
M_3	M_3	$M_2 \oplus M_4$	$M_1 \oplus M_3 \oplus M_5$	$M_2 \oplus M_5^{(2)}$	$M_5^{(3)}$
M_4	M_4	$M_3 \oplus M_5$	$M_2 \oplus M_5^{(2)}$	$M_1 \oplus M_5^{(3)}$	$M_5^{(4)}$
M_5	M_5	$M_5^{(2)}$	$M_5^{(3)}$	$M_5^{(4)}$	$M_5^{(5)}$

The first row and column are clear as M_1 is the tensor-unit and the last row and column follow since the subcategory of projective modules is a tensor-ideal. For the other six entries we use the computer package GAP (see [26]).

First we load the 'QPA' package (see [28]) into GAP, define the quiver Q with one vertex and one arrow and define the algebra $A \cong kG$ which is the path algebra of Q factored out by the relation $x^5 = 0$ where x is the single arrow of the quiver Q.

```
gap> LoadPackage("qpa");
gap> Q:=Quiver(1,[[1,1,"x"]]);
gap> kQ:=PathAlgebra(Field(Z(5)),Q);
gap> AssignGeneratorVariables(kQ);
gap> relations:=[x^5];
gap> A:=kQ/relations;
```

Next we define the five indecomposable variables $M_1, ..., M_5$ in GAP using the following input.

```
gap> M1:=RightModuleOverPathAlgebra(A,[["x",[[0*Z(5)]]]]);
gap> M2:=RightModuleOverPathAlgebra(A,[["x",[[0*Z(5),Z(5)^0],[0*Z(5)
,0*Z(5)]]]]);
gap> M3:=RightModuleOverPathAlgebra(A,[["x",[[0*Z(5),Z(5)^0,0*Z(5)],
[0*Z(5),0*Z(5),Z(5)^0],[0*Z(5),0*Z(5),0*Z(5)]]]]);
gap> M4:=RightModuleOverPathAlgebra(A,[["x",[[0*Z(5),Z(5)^0,0*Z(5),
0*Z(5)],[0*Z(5),0*Z(5),Z(5)^0,0*Z(5)],[0*Z(5),0*Z(5),0*Z(5),Z(5)^0]
,[0*Z(5),0*Z(5),0*Z(5),0*Z(5)]]]]);
gap> M5:=RightModuleOverPathAlgebra(A,[["x",[[0*Z(5),Z(5)^0,0*Z(5),
0*Z(5),0*Z(5)],[0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5)],[0*Z(5),0*Z(5),
0*Z(5),Z(5)^0,0*Z(5)],[0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0],[0*Z(5),
0*Z(5),0*Z(5),0*Z(5),0*Z(5)]]]]);
```

Next we calculate the tensor products by hand. Here M_i has generators $1, T, ..., T^{i-1}$ and $M_i \otimes M_j$ has generators $1 \otimes 1, T \otimes 1, ..., T^{i-1} \otimes 1, ..., 1 \otimes T^{j-1}, ..., T^{i-1} \otimes$

T^{j-1}. We fix the order of the generators and calculate the action of T noting that $T(x \otimes y) = Tx \otimes y + x \otimes Ty + Tx \otimes Ty$ for all generators x and y.

We shall provide working for $M_2 \otimes M_3$. For all other tensor products we will simply display the GAP code. We have generators $1 \otimes 1, T \otimes 1, 1 \otimes T, T \otimes T, 1 \otimes T^2, T \otimes T^2$. Therefore $T(1 \otimes 1) = T \otimes 1 + 1 \otimes T + T \otimes T$ so the first row of our matrix will be $[0, 1, 1, 1, 0, 0]$. Indeed the action of T is given by multiplying on the left by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We define the 6-dimensional module $N := M_2 \otimes M_3$ in GAP using the following input code.

```
gap> v0:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> v234:=[0*Z(5),Z(5)^0,Z(5)^0,Z(5)^0,0*Z(5),0*Z(5)];
gap> v4:=[0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5)];
gap> v456:=[0*Z(5),0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,Z(5)^0];
gap> v6:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0];
gap> mat:=[v234,v4,v456,v6,v6,v0];
gap> N:=RightModuleOverPathAlgebra(A,[["x",mat]]);
```

Using the 'DecomposeModule' command we see that $M_2 \otimes M_3$ decomposes as an indecomposable 2-dimensional module and an indecomposable 4-dimensional module. Since in this example there is only one indecomposable module of each dimension, we can deduce that $M_2 \otimes M_3 \cong M_2 \oplus M_4$. The 'IsDirectSummand' command confirms this conclusion.

```
gap> DecomposeModule(N);
[ <[ 2 ]>, <[ 4 ]> ]
gap> IsDirectSummand(M2,N);
```

```
true
gap> IsDirectSummand(M4,N);
true
```

We define the 4-dimensional module $L := M_2 \otimes M_2$ in GAP using the following input code.

```
gap> u0:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> u4:=[0*Z(5),0*Z(5),0*Z(5),Z(5)^0];
gap> u234:=[0*Z(5),Z(5)^0,Z(5)^0,Z(5)^0];
gap> L:=RightModuleOverPathAlgebra(A,[["x",[u234,u4,u4,u0]]]);
```

Using the 'DecomposeModule' command and the 'IsDirectSummand' command we can see that $M_2 \otimes M_2 \cong M_1 \oplus M_3$.

```
gap> DecomposeModule(L);
[ <[ 1 ]>, <[ 3 ]> ]
gap> IsDirectSummand(M1,L);
true
gap> IsDirectSummand(M3,L);
true
```

We define the 8-dimensional module $H := M_2 \otimes M_4$ in GAP using the following input code.

```
gap> w0:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> w256:=[0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5)]
;
gap> w367:=[0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5)]
;
gap> w8:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0];
gap> w6:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5)];
gap> w7:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5)];
gap> w478:=[0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0]
```

```
;
gap> H:=RightModuleOverPathAlgebra(A,[["x",[w256,w367,w478,w8,w6,w7,
w8,w0]]]);
```

Using the 'DecomposeModule' command and the 'IsDirectSummand' command we can see that $M_2 \otimes M_4 \cong M_3 \oplus M_5$.

```
gap> DecomposeModule(H);
[ <[ 3 ]>, <[ 5 ]> ]
gap> IsDirectSummand(M3,H);
true
gap> IsDirectSummand(M5,H);
true
```

We define the 9-dimensional module $J := M_3 \otimes M_3$ in GAP using the following input code.

```
gap> y0:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5)];
gap> y245:=[0*Z(5),Z(5)^0,0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),0*Z(5),
0*Z(5)];
gap> y356:=[0*Z(5),0*Z(5),Z(5)^0,0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),
0*Z(5)];
gap> y6:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),0*
Z(5)];
gap> y578:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),Z(5)^0,Z(5)^0,
0*Z(5)];
gap> y689:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),Z(5)^0,
Z(5)^0];
gap> y9:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(
5)^0];
gap> y8:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*
Z(5)];
```

```
gap> J:=RightModuleOverPathAlgebra(A,[["x",[y245,y356,y6,y578,y689,y
9,y8,y9,y0]]]);
```

Using the 'DecomposeModule' command and the 'IsDirectSummand' command we can see that $M_3 \otimes M_3 \cong M_1 \oplus M_3 \oplus M_5$.

```
gap> DecomposeModule(J);
[ <[ 1 ]>, <[ 3 ]>, <[ 5 ]> ]
gap> IsDirectSummand(M3,J);
true
gap> IsDirectSummand(M1,J);
true
gap> IsDirectSummand(M5,J);
true
```

We define the 12-dimensional module $W := M_3 \otimes M_4$ in GAP using the following input code.

```
gap> z0:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*
Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> z256:=[0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),
0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> z367:=[0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),
0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> z478:=[0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,
0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> z8:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*
Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> z6910:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5)
,Z(5)^0,Z(5)^0,0*Z(5),0*Z(5)];
gap> z71011:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5
),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5)];
gap> z81112:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^
```

```
0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0];
gap> z12:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),0*Z(5),0*Z(5),Z(5)^0];
gap> z10:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),Z(5)^0,0*Z(5),0*Z(5)];
gap> z11:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),0*Z(5),Z(5)^0,0*Z(5)];
gap> W:=RightModuleOverPathAlgebra(A,[["x",[z256,z367,z478,z8,z6910,
z71011,z81112,z12,z10,z11,z12,z0]]]);
```

Using the ‘DecomposeModule’ command and the ‘IsDirectSummand’ command we can see that $M_3 \otimes M_4 \cong M_2 \oplus M_5 \oplus M_5$.

```
gap> DecomposeModule(W);
[ <[ 2 ]>, <[ 5 ]>, <[ 5 ]> ]
gap> IsDirectSummand(M5,W);
true
gap> IsDirectSummand(M2,W);
true
```

We define the 16-dimensional module $F := M_4 \otimes M_4$ in GAP using the following input code.

```
gap> s0:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*
Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s256:=[0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),
0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s367:=[0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),
0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s478:=[0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,
0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s8:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*
Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
```

```
gap> s6910:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5)
,Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s71011:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5
),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s81112:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^
0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s12:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),0*Z(5),0*Z(5)];
gap> s101314:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(
5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5),0*Z(5)];
gap> s111516:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(
5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),0*Z(5),Z(5)^0,Z(5)^0];
gap> s121516:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(
5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0];
gap> s111415:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(
5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5),Z(5)^0,Z(5)^0,0*Z(5)];
gap> s16:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0];
gap> s14:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5),0*Z(5)];
gap> s15:=[0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0
*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),0*Z(5),Z(5)^0,0*Z(5)];
gap> F:=RightModuleOverPathAlgebra(A,[["x",[s256,s367,s478,s8,s6910,
s71011,s81112,s12,s101314,s111415,s121516,s16,s14,s15,s16,s0]]]);
```

Using the 'DecomposeModule' command we can see that $M_4 \otimes M_4 \cong M_1 \oplus M_5 \oplus M_5 \oplus M_5$.

```
gap> DecomposeModule(F);
[ <[ 1 ]>, <[ 5 ]>, <[ 5 ]>, <[ 5 ]> ]
```